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Multi-horizon portfolio insurance model

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Abstract

This paper presents and analyses the structure of a new mutual fund management model that allocates a fund's resources with respect to the different time preferences of investors. The proposed model enables the explicit definition of investment horizons in a regular open-ended fund framework that uses the popular portfolio insurance strategy based on value at risk. Investors are assumed to be homogeneous in terms of their risk/reward preferences but heterogeneous in terms of their investment horizons. Time moments when investment decisions are made by individual investors are spread out over time randomly because of the different life cycles of investors. We assume that all investors in the fund can be separated into manageable numbers of groups regarding their remaining investment horizons. The fundamental concept of the proposed multi-horizon portfolio insurance model is optimising the composition of the fund according to the most conservative allocation among the optimal portfolios of all considered groups of investors. A historical simulation based on US financial data is also used to compare the proposed model with the regular single horizon strategy and to stress test proposed model with its various parameterisations.

Keywords

Multi-investment horizons, portfolio choice, portfolio insurance, Value at Risk.

JEL Classification: G11, G23

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1. Introduction

The fundamental objective of most investment decisions made by individual investors is to capitalise on invested assets. The time horizon, during which expected capitalisation is supposed to materialise, is an essential attribute of investment decisions. Solving the optimal portfolio selection problem encompasses the specification of an opportunity set and definition of a preference or evaluation function.

Let's define the opportunity set of an individual investor as a universe of tradable financial assets, particularly a mix of mutual funds that spans the entire efficient frontier. Furthermore, let's assume that individual investors make correct investment decisions and select an optimal mix of funds. However, the time moments within which investment decisions are made by individual investors are spread out over time randomly because of the different life cycles of investors. Investors that invest in particular funds are thus homogeneous in terms of their risk/reward preferences but heterogeneous in terms of their investment horizons and reference moments of performance attribution.

The investment strategies of mutual funds differ in terms of the strength of commitment to fulfil the investment horizon objectives. The vast majority of open-ended funds are offered with an implicitly defined investment horizon, which is usually defined as a time interval. All closed-ended funds and the minority of open-ended funds are stated in terms of an explicitly defined investment horizon. The investment strategies of implicitly defined investment horizon mutual funds focus exclusively on the length of the investment period and do not tie investment decisions to concrete time moments. The investment horizon is continuously rolled over. A fund's portfolio is thus adjusted to fulfil the preferences of those investors that have just invested. Let's materialise investment decisions into a distribution of risk/reward underlying factors. This distribution can be provided as an expert assessment (some sort of risk/reward characteristics), by the use of statistical estimators or a Bayesian mix of both. Statistical estimates are required to be unbiased and efficient and robust to foreseeable future events. If the estimators were absolutely unbiased, the investment horizon objectives would be fulfilled on average. However, there exist unsatisfied investors because of the presence of inefficiency. This is the main reason why most open-ended mutual funds offer implicitly defined investment horizons. The investment strategies of these explicitly defined investment horizon mutual funds usually exploit investors' preferences for investment products that limit maximum losses. Popular portfolio insurance strategies, life cycle funds or held to maturity bond strategies are examples of such explicitly defined investment horizon strategies.

In addition to the objective of maximising future value, individual investors usually prefer investments that limit potential losses. This behaviour is described by behavioural finance and formalised under the cumulative prospect theory model developed by Kahneman and Tversky (1992). It has been shown that investors search for downside protection strategies because of skewed probability weighting, loss aversion and the S-shaped curvature of the utility function. Portfolio insurance strategies are constructed to maximise invested capital while limiting maximum losses within explicitly defined investment horizons.

The objective of the present article is to introduce and analyse the structure of a new mutual fund management model that enables the explicit definition of investment horizons in regular open-ended fund frameworks. We focus on the structure of the proposed model and leave the search for the best parameterisation to future research. Firstly, investors of particular funds are divided into separate subgroups according to their moments of investment. There is a unique investment period for every subgroup that is bounded by the moment of investment and the investment horizon. The proposed model focuses on all active investment periods. Secondly, a dynamic value-at-risk (VaR)-based portfolio insurance (VBPI) strategy is chosen as an optimisation tool because of its flexibility in parameterisation and with the elaborated framework. However, the constructed model is not restricted to using VBPI only. An investor's expected utility or index of satisfaction is assumed to be maximised by portfolio insurance strategies. The primary concept of the proposed model is to construct an optimal portfolio for each active investment period. A fund's portfolio is allocated correspondingly to the most conservative optimal portfolio. If there exist only two assets, namely risk-free and risky assets, and the efficient frontier is strictly increasing, the most conservative portfolio is defined as that with the lesser allocation of risky assets.

Regular single horizon VBPI is first examined in general. Then, the two-asset case that enables a closed-ended analytical solution is emphasised. The next section formalises multi-horizon VBPI (MH-VBPI) in detail to show how and when opportunity costs arise for some investors of the fund managed by MH-VBPI. Historical back-testing elaborates on the differences between the single horizon and multi-horizon strategies and shows under what circumstances both approaches are identical. Finally, the effects of different investment horizons, minimum performance requirements and the confidence level of the α-quantile estimator are analysed within a historical simulation setting.

2. Value-at-Risk-based portfolio insurance

The concept of maximising expected portfolio value while controlling for a shortfall probability is a conventional alternative to the mean/variance framework. This idea was first suggested by Telser (see Elton et al., 2003) as a single period evaluation function and developed by Leibowitz and Kogelman (1991) into a multi-period portfolio insurance strategy. The constant proportion portfolio insurance (CPPI) model was introduced by Perold (1986) on fixed income assets. Black and Jones (1987) extended this method by using equity-based underlying assets. Goetzmann and Broadie (1992) introduced the safety-first portfolio insurance program, thus improving on the popular CPPI model and time-invariant portfolio protection by directly controlling for shortfall probability. The safety-first portfolio insurance program based on Telser's safety-first criterion is identical to the VBPI strategy proposed by Chow and Kritzman (2001) and Herold et al. (2005). Both approaches utilise the estimated quantile of future portfolio value distribution as a risk measure and dynamically rebalance portfolio composition. Hamidi et al. (2009) elaborated on a closely related approach, namely time varying proportion portfolio insurance. Fruitful studies from behavioural finance have proven that the safety-first criterion is consistent with the way investors perceive risk (Atwood et al., 1988; Harlow, 1991; Brogan and Stidham, 2005). VBPI can be classified as a dynamic hedging method set to hedge total risk, defined as an undesirable performance.

2.1 General case

VBPI can be formalised as the following constrained optimisation problem on portfolio percentage weights' vector **w**:

$$\max_{w} E(V_T) \tag{1}$$

subject to
$$w \in C, \tag{2}$$
$$P(V_T \le CV) = \alpha \tag{3}$$

$E(V_T)$ represents an expected portfolio value for a given investment horizon *T*, calculated in *t*. The time moment *t* can be interpreted as the moment when the investment decision is made. The length of the remaining investment period is defined as $\tau = T - t$. *C* is a suitable set of investment constraints. *CV* (critical value) is the minimal portfolio value required at *T* where a small probability equal to α exists that the true future portfolio value will be less than *CV*. The minimal portfolio value constraint (3) is described as a generic probability measure $P(\ )$. It is assumed that all expenses incurred by an investor are included in the calculation of expected value, quantile and VaR.

Let's define $Q_\alpha(V_T)$ as an α-quantile of the expected portfolio value. The corresponding VaR of a portfolio's returns then holds:

$$VaR_\alpha(R_{T,\tau}) = -\left(\frac{Q_\alpha(V_T)}{V_t} - 1\right), \tag{4}$$

where $R_{T,\tau}$ is the linear return of portfolio value between the time moments *t* and *T* and V_t equals the portfolio value at *t*. The constraint (3) can be thus restated as:

$$Q_\alpha(V_T) \ge CV \tag{5}$$

or equivalently as:

$$VaR_\alpha(R_{T,\tau}) \le \left(\frac{CV}{V_T} - 1\right)(-1). \tag{6}$$

The stochastic character of the returns on financial assets causes a fluctuation in the interim portfolio value and this could lead to a change in expectations. As a result, the portfolio's composition must be rebalanced. Let's specify the moment of an investment more precisely as t_0 and the moment of rebalancing as t_1. The rebalancing is formalised as:

$$\max_{w} E(V_T)$$

subject to
$$w \in C,$$
$$Q_\alpha(V_T) \ge CV.$$

The investment horizon of *T* is unchanged and the realisations of portfolio returns between (t_0, t_1) determine the portfolio value at t_1. Dynamically managed portfolios can be rebalanced based on several methods. Jiang et al. (2009) presented three disciplines: time discipline, market move discipline and portfolio mix discipline. The first requires rebalancing the portfolio at predetermined time intervals; the second when a pre-specified move is realised in the market; and the last when the difference between the required and current portfolio composition exceeds a specified threshold. The inevitable consequences of rebalancing are transaction costs. A detailed description of transaction costs and their inclusion in the optimisation problem is given by Fabozzi et al. (2006). VBPI is closely linked to another portfolio insurance strategy, namely the CPPI model.

2.2 Two-asset model

Let's restrict analysis further to the presence of two financial assets only, namely risk-free and risky assets, which are independent by definition. Risky assets can be interpreted in the spirit of modern portfolio theory as a market portfolio. All combinations of risk-free and risky assets span an efficient frontier, namely the capital market line. The efficient frontier strictly increases while the expected return on risky assets is higher than is the risk-free return and for all pairs of efficient portfolios P_A, P_B holds:

$$E(P_A) > E(P_B) \rightarrow Q(P_A) > Q(P_B),$$

where $Q(\ )$ represents the risk measure. In this setting, condition (5) is sufficient to determine the optimal portfolio that satisfies the minimal portfolio value requirement. The investor specifies his/her risk budget with respect to the chosen minimal portfolio value and the current portfolio value and invests in the risk-free and risky assets such that the α-quantile of the expected portfolio value equals the minimal portfolio value, satisfying all other constraints.

Let's assume that the underlying factors of both risk-free and risky assets are continuously compounded returns $\boldsymbol{r}$ and $\boldsymbol{c}$.[1] Under these assumptions, VBPI can be reformulated to search for the proportion of risk-free and risky assets such that the α-quantile of compounded returns on the portfolio holds:

$$Q_\alpha\left(c^P_{T,\tau}\right) = \ln\left(\frac{CV}{V_T}\right). \tag{7}$$

The α-quantile of portfolio returns $Q_\alpha\left(c^P_{T,\tau}\right)$ is determined by the proportion of risk-free and risky assets and the probability distribution function of the compounded returns on risky assets. Let's assume that a budget constraint is desired. The proportion of risky asset equals w_t and the proportion of risk-free assets is $(1 - w_t)$. The condition (5) can thus be rewritten as:

$$w_t V_t \exp\left(Q_\alpha\left(c^{RA}_{T,\tau}\right)\right) + (1 - w_t) V_t \exp(r\ \tau) = CV. \tag{8}$$

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[1] The future value of financial asset *A*, which is fully described by continuously compounded returns $\boldsymbol{c}$, is given as: $A_T = A_t \exp(c_{T,\tau})$, where $c_{T,\tau}$ is the realisation of the return $\boldsymbol{c}$ in time interval (t, T) and $\tau = T - t$.

$Q_\alpha\left(c_{T,\tau}^{RA}\right)$ is an α-quantile of the continuously compounded returns on risky assets, r is the continuously compounded returns on risk-free assets defined in the same basis as τ (e.g. per annum), t is the moment of portfolio creation or rebalancing and T is the investment horizon. Rearranging (8) gives the analytic solution for the proportion of risky assets:

$$w_t = \frac{CV - V_t\ \exp(r\ \tau)}{V_t\ \exp\left(Q_\alpha\left(c_{T,\tau}^{RA}\right)\right) - V_t\ \exp(r\ \tau)}. \tag{9}$$

The performance of VBPI is driven by the ability to correctly estimate the future probability distribution of the returns on risky assets and to manage the inefficiency of the estimates, e.g. by rebalancing the policy and treatment of risk-free asset returns, which is briefly introduced in Appendix B. Equation (9) can then be reformulated to find out its economical interpretation:

$$w_t = \left(\frac{\exp(r\ \tau)}{V_t}\right)\frac{V_t - CV\exp(-r\ \tau)}{\exp(r\ \tau) - \exp\left(Q_\alpha\left(c_{T,\tau}^{RA}\right)\right)}. \tag{10}$$

The numerator of the second fraction $\{V_t - CV\exp(-r\ \tau)\}$ represents the risk budget of the portfolio. The risk budget is the present value of the portfolio that can be lost when risky assets decrease. More often, this is labelled *cushion* and is the focus for the direct calculation of exposure to risky assets in CPPI models. The denominator of the second fraction $\left\{\exp(r\ \tau) - \exp\left(Q_\alpha\left(c_{T,\tau}^{RA}\right)\right)\right\}$ is the return differential between the risk-free return and the worst considered return on risky assets. Appendix A elaborates more on the analytics of VBPI models and interconnections with CPPI models.

3. Multi-horizon VBPI models

Consider an open-ended mutual fund that aims to fulfil the objectives of investors with explicitly defined investment horizons. These investors are risk-averse and anticipate risk as a return (or future portfolio value) that is lower than is the pre-specified critical value. Furthermore, their utility functions are optimised by portfolio insurance strategies. We assume that investors are homogeneous in terms of their risk/reward preferences but heterogeneous in terms of their investment horizons and the reference moments of performance attribution because of different life cycles. All investments are made in discrete time.

Because the total number of all considered investors can be unmanageably large, we first split them up into separate groups with respect to different investment periods. Each investment period is determined by a unique beginning date, critical value and moment of investment horizon maturity. The portfolio value is required to reach the critical value at this time moment. Let's define the number of different investment periods that begin during one calendar year as frequency γ and recall the length of investment period to be τ. Hence, there exist Y investment periods that run in every time moment t:

$$Y = \tau\gamma. \tag{11}$$

Furthermore, we mark the remaining time to maturity of period y as τ^y. The analysis is constrained to the described case of two assets, namely risk-free and risky assets that span the entire adjusted efficient frontier where the dimension of standard deviation is substituted for the dimension of VaR. The risk-free asset yields continuously compounded rate r and the risk-free yield curve is assumed to be flat. The distribution of the risky asset's prices is mapped to the distribution of underlying factors. We consider the simpler pricing function where underlying factors are compounded returns. The optimal portfolio related to the investment period y is calculated in t following (9) and is determined by CV_y, V_t and $\hat{Q}_\alpha^y\left(c_{T,\tau}^{RA}\right)$. We also define vector $\mathbf{v}$, which contains all estimated α-quantiles made in t, where every estimate relates to a different investment period, and thus to a different time moment:

$$\mathbf{v} = \left\{\hat{Q}_\alpha^y(\ )\right\}_{y=1,\ldots,Y}. \tag{12}$$

The total number of quantile estimates is equal to the total number of investment periods Y, and vector $\mathbf{v}$ can be interpreted as a time structure of risk. Alternatively, the quantile estimation can be performed for the generic estimation interval $\tilde{\tau}$ (e.g. a time series analysis of weekly returns) and subsequently projected to the required investment horizons. Time projection can be defined as a problem of projecting the distributional characteristics of financial underlying factors from an estimation interval to any other point in time. More formally, the time projection function can be defined as:

$$\theta^y = f^y\left(\hat{\theta}\right). \tag{13}$$

This projects parameter $\hat{\theta}$ estimated in the generic estimation interval to the required time moment $t + \tau^y$. A particular form of the projection function depends on the assumed process of security prices and underlying factors.[2] Interested readers are referred to

[2] Three broad groups of time projection functions can be specified with regard to the underlying process assumptions: identically and independently distributed continuously compounded returns; dynamic volatility models; and alternative hypotheses. The time projection function under the i.i.d. assumption within an elliptical class of distribution is analysed in standard financial textbooks. The projection of higher statistical moments under the i.i.d. process assumption was examined by Meucci (2004, 2010) and Duc and

Meucci (2004) for a formal definition of time projection. The term structure of risk is treated by Guidolin and Timmermann (2006); Colacito and Engle (2009) and Brownlees et al. (2009).

Let's define the vector $\boldsymbol{\zeta}$, that consists of all Y critical values:

$$\boldsymbol{\zeta} = \{CV_y\}_{y=1,\dots,Y}. \tag{14}$$

The value of a particular CV_y is time invariant because its value is defined at the beginning of the investment period. Finally, the composition of the optimal portfolio is calculated following (9) for every investment period. The composition of the optimal portfolio is fully determined by the proportion of the risky asset w_t as we assume a two-asset case and impose budget constraints. The optimal proportions of risky assets calculated in t can be aggregated to the vector **w**:

$$\mathbf{w} = f\left(\nu, \zeta, f^{y}(\ )\right). \tag{15}$$

The structure of the vector **w** determines the final composition of the considered mutual fund. As the fund's objective is to fulfil the preferences of all investment periods, the allocation to risky assets is set as a minimum of all components of **w**:

$$w_t^{MH-VBPI} = \min(\mathbf{w}). \tag{16}$$

Equation (16) can be interpreted as a particular choice of decision function that finds a fund's allocation. The proposed model follows the allocation of investment period most at risk at every point of time. However, the investment period that triggers this allocation changes dynamically with respect to market developments. Most often, a binding constraint will be imposed by a longest investment period. If a negative return is realised, a multi-horizon strategy will be driven by shorter investment periods. The strategy is suboptimal for new investors in this case. By contrast, the opportunity costs are lower than are those of a regular portfolio insurance strategy, which is often suboptimal for new investors until the final maturity date of the strategy, or while the original critical value suits the requirements of new investors. Additionally, it can be assumed that the mechanism of multiple binding constraints reduces estimation risk.

The fund's allocation $w_t^{MH-VBPI}$ is not restricted to being driven by the minimum optimal allocation of various investment periods. In a multivariate setting, a more elaborate decision function must be used. Alternatively, Hamidi et al. (2009) suggested using the average as the decision function in order to reduce the start date and horizon date bias in their version of VBPI.

3.1 Example calculations

A simplified example is presented in the following paragraph. Funds' prices are illustrated to demonstrate the functioning of the model. We assume that there are four two-year investment periods and that a new investment period begins every six months. Investors require at least 98% of invested capital at the investment horizon. Thus, the critical value is set to 98% of the fund's price at the beginning of a particular period. Table 1 shows all the required inputs and calculations at time t_1. The length of the investment period is equal to two years for every period. The remaining time to maturity is longest for *Period 4*, which is assumed to be the period that has just begun. The oldest period, *Period 1,* matures in six months. The next columns show the prices of the fund at the beginning of particular periods and its corresponding critical values. The current fund's price equals 105.0 (which is the reference for the calculation of *Period 4*'s critical value). We assume that the fund's price has increased by 5% since the beginning of *Period 1*. The columns labelled r_f and RB_y contain the compounded risk-free rates that are assumed to be gained in the remaining time to maturity of the investment period and particular risk budget. The risk budgets are calculated from (10) as the difference between the current fund's price (105.0 for every period) and the present value of a particular critical value, and these are expressed as a percentage of the current fund's price. The risk budget is highest for *Period 1* as we have assumed a 5% performance of the fund in the past 1.5 years. The lowest risk budget is achieved by *Period 4* as the fund performed well since the beginning of *Period 2* and *Period 3*. The next column shows the estimated term structure of risk for the remaining time to maturi-

Schorderet (2008). We refer interested readers to Engle (2009) for the treatment of dynamic volatility models and consequent time projection as well as for further references. The alternative hypothesis is postulated and analysed in Pástor and Stambaugh (2009). A broad class of scenario analysis can be attributed to alternative hypotheses, as well.

Table 1 MH-VBPI example calculation at t_1

	τ	τ^y	V_y	CV_y	r_f	RB_y	*VaR* (1%)	**w** (risky asset)
Period 1	2	0.5	100.0	98.0	0.9%	7.9%	11.7%	63.6%
Period 2	2	1.0	102.0	100.0	1.8%	6.7%	15.6%	40.7%
Period 3	2	1.5	103.4	101.3	2.7%	6.1%	18.3%	32.0%
Period 4	2	2.0	105.0	102.9	3.6%	5.4%	20.3%	**25.7%**

ties in all periods, defined as VaR figures at the 1% confidence level.[3] The optimal allocations for all periods are calculated following (9) and are shown in the last column. The fund's composition is driven by the optimal allocation of *Period 4* because of positive performance in proceeding periods and the chosen term structure of the risk.

We also assume that the fund lost approximately 3.8% in the following six months[4] and that its price reached 101.0. The current fund's price represents the basis for the calculation of the critical value of the newest investment period. *Period 2* will now mature in six months and the newest period is labelled *Period 5*. Table 2 shows all the statistics at time t_2. We assume in our example that risk-free rates as well as the term structure of risk remained unchanged despite the decreasing value of risky assets.

The risk budget of *Period 4* is now the lowest, as the difference between the current fund's price (101.0) and the present value of *Period 4*'s critical value (102.9) is the lowest. Using (9) and minimising all assumed investment periods yields 4.4% risky assets. The binding constraint is shifted from the longest investment period to the period most at risk.

4. Multi-Horizon VBPI Performance

Two questions are particularly important here. Firstly, how different is the performance of the presented multi-horizon portfolio insurance model from that of the regular single horizon strategy when both utilise the same portfolio insurance optimisations. Secondly, how sensitive is the multi-horizon strategy to different investment horizons, critical values and the parameters of the used α-quantile estimator. Thus, a historical simulation based on weekly US financial data from 5.1.1962 to 22.10.2010 to elaborate on both issues was used.

Risky assets were calculated as a 40/60 mix of equities and long-term government bonds. Equities were represented by the S&P500 index with reinvested dividends[5] and we used the total return index of generic 10-year US Treasury bonds as a proxy for long-term bonds.[6] The composition of risky assets was rebalanced weekly and we assumed nil transaction costs. The risk-free asset was represented by the total return index composed of US Treasury bills with 12-month maturity dates. We used the current yield of 12-month Treasury bills expressed as a continuously compounded rate (risk-free rate).

We assumed that the compounded returns on risky assets showed a conditional normal distribution in order to keep calculations as simple as possible. However, the usage of biased and inefficient estimators of the α-quantile did not influence our analysis as both MH-VBPI and VBPI work with the same optimisations. Standard deviation was estimated as a sample estimate from a rolling window of 26 past weekly observations and the expected return on risky assets was held constant at 3% in all simulations.[7] The time projection of estimated parameters in different investment horizons was carried out following the square root rule of time.

4.1 Comparison to a single horizon strategy

In this part, we simulate the performances of hypothetical mutual funds managed by various MH-VBPI and VBPI models over two-year investment horizons. All portfolios are rebalanced weekly, negative risky asset weight is restricted and the confidence level equals 1%. The multi-horizon model is composed of 12 investment periods that run in every time moment. A new investment period thus starts every two

[3] We assume a normal distribution of a risky asset's compounded returns with an expected value of 0.03 and a standard deviation of 0.08.

[4] The loss of 3.8% at the fund level corresponds to a slump of approximately 17.4% in the value of risky assets, as 25.7% of the fund's assets were invested in risky assets at t_1.

[5] The S&P500 total return index was used since its inception in 1988. We used the average dividend yield calculated from January 4, 1988 to October 22, 2010 to correct older data.

[6] The index composition was recalculated every month. The nine-year and 11-month US Treasury bond was assumed to be sold and a new 10-year bond was included in the index, both at market prices.

[7] We do not use the time varying risk-free rate of return as a proxy for a risky asset's expected return because of its procyclical character. The risk-free rate decreases in downturn markets as a result of easier monetary policy and thus reduces the risk budget unnecessarily. On the contrary, a high risk-free rate enables too risky positions in upmarkets.

Table 2 MH-VBPI example calculation at t_2

	τ	τ^y	V_y	CV_y	r_f	RB_y	*VaR* (1%)	**w** (risky asset)
Period 2	2	0.5	102.0	100.0	0.9%	1.9%	11.7%	16.2%
Period 3	2	1.0	103.4	101.3	1.8%	1.4%	15.6%	9.1%
Period 4	2	1.5	105.0	102.9	2.7%	0.8%	18.3%	**4.4%**
Period 5	2	2.0	101.0	99.0	3.6%	5.4%	20.3%	25.7%

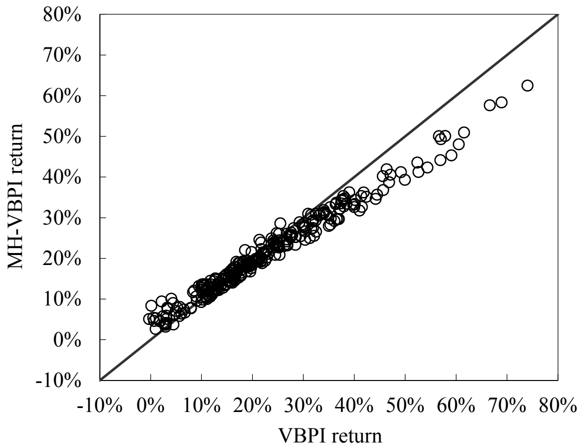

Figure 1 CV = 95%, Max Leverage = 100%

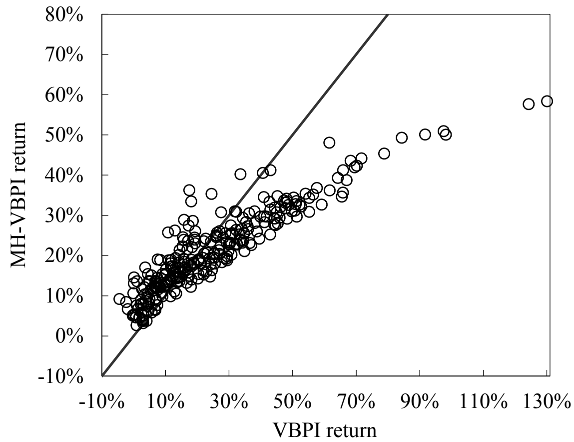

Figure 2 CV = 100%, Max Leverage = 100%

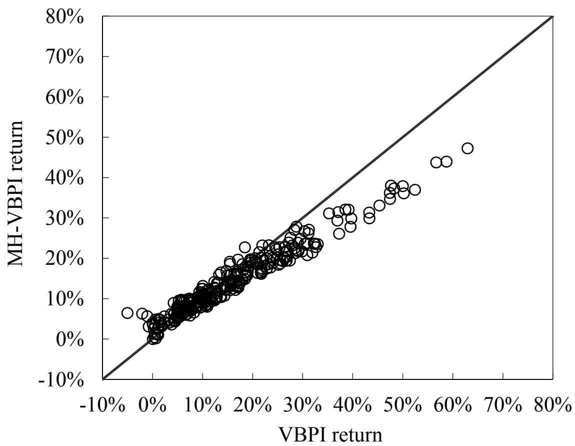

Figure 3 CV = 95%, Max Leverage = 300%

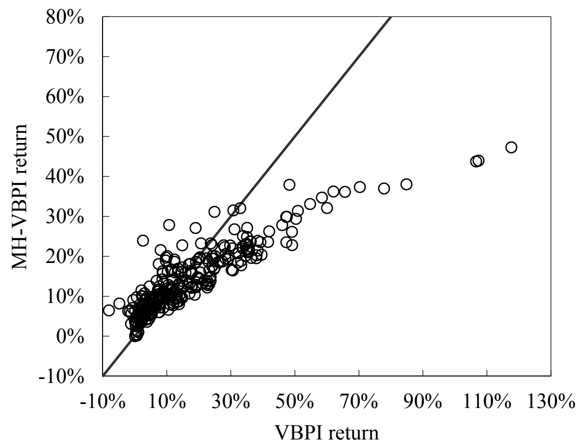

Figure 4 CV = 100%, Max Leverage = 300%

months. A regular single horizon strategy is simulated for every investment period considered in MH-VBPI. Figures 1–4 show the back-tested returns on MH-VBPI in relation to the returns on VBPI with respect to different critical values and the maximum leverages of risky assets. Each circle represents the returns on both strategies realised in specific two-year investment periods. The *x* axis shows VBPI returns and *y* axis returns on MH-VBPI.

MH-VBPI performs closely to VBPI in most cases. As expected, a multi-horizon strategy realises lower returns than does a regular strategy in high return spaces. This is the result of new critical value inclusion that is delivered periodically. As a result, the convex character of MH-VBPI is restricted. The return differential in high return spaces increases when we allow for risky asset leverage. VBPI exploits higher leverage, whereas a multi-horizon framework remains unchanged. By contrast, higher returns on VBPI are paid with lower minimum returns and left tail percentiles. VBPI violates critical value requirements in all cases apart from the simulation shown in Figure 1. Higher maximum leverage helps MH-VBPI outperform in lower return spaces, as higher leverage overexposes VBPI to inefficient estimators and market volatility.

Figure 5 gives other insights into the differences between MH-VBPI and VBPI. Funds managed by MH-VBPI can be seen as portfolios of several single horizon VBPIs. Allocation into a particular VBPI is mutually exclusive as MH-VBPI invests solely into the most conservative alternative. This is obviously desirable when the most conservative allocation is induced by the newest investment period with the longest horizon. Otherwise, there arise opportunity costs incurred by investors that buy the fund subsequently. However, rolling explicitly defined horizons and possibly lower estimation errors offset those costs. Figure 5 shows the average composition of considered funds in the period from 5.1.1962 to 22.10.2010 with respect to various critical values. Individual components are single horizon VBPIs. The investment horizon is two years for both MH-VBPI and individual VBPIs. We assume 12 VBPIs running at every moment. The confidence level is kept at 1%. The VBPI labelled as period 1 is the strategy with the shortest horizon, while the VBPI labelled as period 12 is the newest strategy.

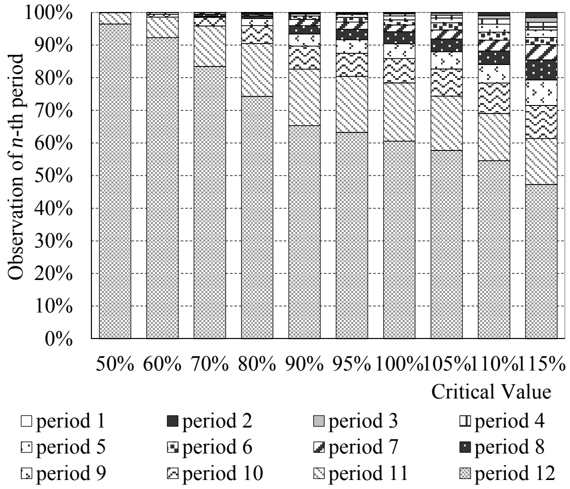

Figure 5 Average composition of MH-VBPI

The fund is mostly invested following the most recent single horizon strategy as desired. However, the horizon strategy agrees with the newest VBPI mostly for risky parameterisations, namely those with lower critical values. Both strategies are identical in more than 95% of cases with a 50% critical value. Here, the risk budget is closely unbounded and the regular inclusion of new VBPI influences MH-VBPI to a small extent. The portfolio of MH-VBPI differs as the critical value increases. Tight risk budgets force a multi-horizon strategy to swiftly change focus period after market shocks in favour of older VBPIs. The percentage of MH-VBPI and newest VBPI concordance continually decreases and both strategies agree approximately 45% of the time for a 115% critical value.

4.2 Sensitivity analysis

We also back-tested the performances of hypothetical mutual funds to assess the sensitivity of MH-VBPI to different investment horizons, critical values and α-quantile estimators. We considered four different investment horizons: one, two, four and six years (labelled *S1Y*, *S2Y*, *S4Y* and *S6Y*). Twelve investment periods run in every moment. In the case of a one-year horizon, a new investment period begins every month. In the case of a six-year horizon, a new period starts semi-annually. The fund's portfolio was rebalanced weekly as in the preceding analysis. Every considered parameterisation utilised the same risk-free and risky assets as well as the risk-free rate and volatility estimate. No leverage of risky assets was allowed.

Investment horizon

The investment horizon determines the composition of the fund through the amount of risk budget, which is defined as the difference between the fund's price and the present value of CV. Additionally, investment horizon or time to maturity enters the time projection of the α-quantile estimate. A longer investment horizon results in a more dynamic allocation as more sources can be allocated to risky assets. Figure 6 compares the selected performance characteristics of the considered strategies in the historical simulation. The critical value was kept at 98% and confidence level at 5%. There were 574 one-year return realisations in the case of a one-year horizon strategy; 281 realisations of two-year returns in the case of a two-year horizon strategy; and 135 and 86 realisations of four- and six-year returns for four- and six-year horizon strategies. The extent of realised returns results from the historical performances of the chosen assets.

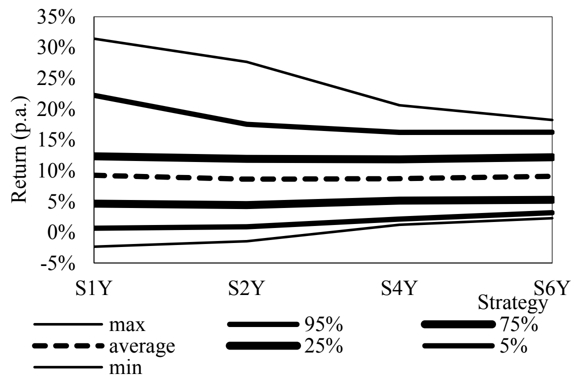

Figure 6 Selected characteristics of strategies' returns

All strategies realised comparable average returns as well as 25th and 75th percentile returns. However, the return distribution of the strategy with the shortest investment horizon is most volatile and presents positive skewness. As expected, both the volatility of realised returns and its skewness decreased for longer horizons. As will be briefly shown, comparable average returns would diminish if transaction costs were contained. To investigate the further risk/reward characteristics of MH-VBPI, Figure 7 shows the realised sample volatilities of the considered strategies' weekly returns with respect to different critical values. The confidence level was kept at 5%.

The highest volatility was realised by the longest horizon strategy. Strategies with shorter horizons realised lower volatilities because of a lower proportion of risky assets. The same holds for the relation of realised volatilities and critical values. Higher critical values lead to more conservative and thus less volatile portfolio performances. However, this relation varies across model specifications. The volatility of the shortest strategy's returns decreases most quickly. On the contrary, the shape of the realised volatility of the strategy with a six-year horizon is relatively horizontal. This is the result of using the same critical values for all models. The decrease in the realised volatility of a six-year strategy would be obvious if values larger than 110% were used as the critical value. The comparison of considered strategies with respect to the average weight of equity allocation shows the same results.

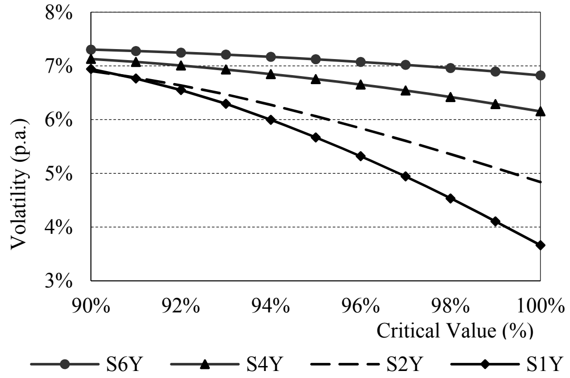

Figure 7 Realised volatilities of the MH-VBPI strategies

Average turnover

Another insight gives a comparison of average weekly equity allocation turnover.[8] Because we assumed nil transaction costs in back-testing, this comparison indicates the possible impact of their inclusion. Figure 8 shows the comparison of equity turnover with respect to different critical values. As in Figure 6, the confidence level was kept unchanged at 5%.

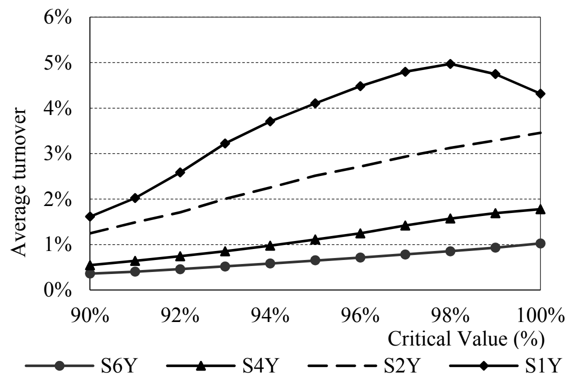

Figure 8 Average weekly turnover of equity allocation

The highest average turnover was realised for the strategy with the shortest considered horizon and for conservative critical values. This is the consequence of the same risky assets that resulted in rather high trading in the shortest strategy because its composition is more sensitive to volatile assets. If we included transaction costs in the simulation, the performance of the shortest strategy would significantly deteriorate. For this strategy, it would be optimal to decrease the equity component of risky assets. Figure 8 shows another interesting relationship between critical value and average turnover as well. The highest turnovers confront strategies that work with high critical values. The risk budget is limited, and even small moves in the prices of risky assets trigger a rebalancing of portfolios.

Critical value

The required critical value has a direct negative relationship with risky asset exposure in single period portfolio insurance models. The lower is the required critical value the higher is the risk budget and portfolio can take larger levels of risky asset exposure. This relationship is expected to hold in multi-horizon models to lesser extent. Figure 9 shows the performance characteristics of the chosen one-year horizon strategy with respect to different required critical values. The confidence level is kept constant at 5%.

The most volatile characteristics are realised for lower values of the required critical values. As expected, higher average returns are delivered by those critical values and the negative relationship between average returns and critical values is obvious as well. Interestingly, the inner quartile interval is not significantly impacted by changing the critical value requirements. The requirements of minimal return were violated for higher critical values only. For CV = 100%, 13 one-year periods were realised with returns lower than zero. As the strategy is performed on an overlapping basis, those periods were caused by a lower number of unanticipated events (such as the 1987 market crash). The other parameterisations of MH-VBPI brought the same results.

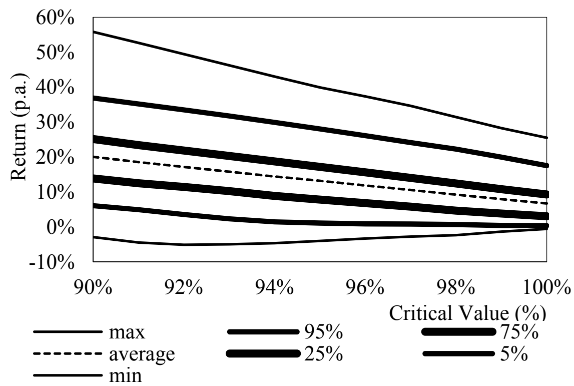

Figure 9 Selected return characteristics of S1Y

α-quantile estimator

In general, two methods of α-quantile estimation are used in portfolio insurance models with regard to forecasting horizons. Hamidi et al. (2009) specified the forecasting horizon as the upper limit of no-trade

[8] We sum the absolute values of equity allocation changes; thus, both positive and negative changes of equity allocation are accounted for.

time interval. The task is to estimate the maximum anticipated loss of risky assets that can be accommodated by the portfolio within this interval. The forecasting horizon is thus much shorter than is the investment horizon of the fund's investors. Alternatively, the forecasting horizon is set to equal the targeted investment horizon as in Herold et al. (2005). In this way, risk is anticipated directly in terms of the characteristics of terminal value and thus long-term risks are acknowledged as well (for more on the intersection between short-term and long-term risks, refer to Engle, 2009). If the investment horizon were about to terminate, both methods would yield the same results. In back-testing, the forecasting horizon was equal to the investment horizon. We used a simple model of conditional normally distributed compounded returns and the α-quantile was calculated from sample volatility, a proxy for expected returns and confidence level. Figure 10 shows the performance characteristics of a one-year horizon strategy for a critical value set to 95% with respect to different confidence levels.

The results shown in Figure 10 are somewhat counterintuitive. The confidence level does not influence average returns of one-year investment periods. However, the shape of the return distribution changes rapidly when for higher confidence levels the probability mass is centred significantly on the average. Even the analysis of the parameterisations of different critical values yields the same results. A higher confidence level significantly reduces the dispersion of realised returns, although the average return is not drastically influenced. The same conclusions can be drawn from the analysis of return sensitivity to different values of constants used as a proxy for expected returns.

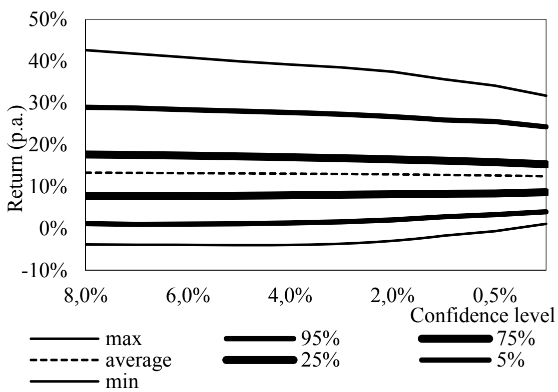

Figure 10 Return characteristics of S1Y

5. Conclusion

In the present paper, a MH-VBPI model that enables an explicit definition of all investors' investment horizons in regular open-ended fund frameworks was defined and analysed. The fundamental concept of the proposed multi-horizon model is to allocate a fund's assets according to the most conservative allocation among optimal portfolios of all groups of investors. VBPI was utilised as the optimisation tool. Obviously, other portfolio insurance optimisers can be used as well.

A historical simulation based on US market data was used to elaborate on (*i*) the differences between proposed MH-VBPI and regular single horizon VBPI and (*ii*) the sensitivities of the proposed model to different parameterisations. A multi-horizon strategy was performed in close relation to the single horizon strategy when excessive leverage was prohibited. The characteristics of the single horizon portfolio insurance model were mostly preserved in the multi-horizon framework. The longer the investment horizon and the lower the required minimal portfolio value, the higher the fund's performance. The composition of risky assets plays an important role when assessing the influence of unwanted trading and thus the same composition cannot be recommended for strategies with different horizons. Interestingly, the value of the confidence level was found to be an important driver of the tails of realised returns, with a rather small effect on average returns.

This proposed research can be variously expanded. Firstly, more advanced simulation methods are required to confirm the presented results. Secondly, coherent risk measures can be used instead of VaR, because they offer better statistical properties, most importantly in multivariate settings. Lastly, there exists a vast range of risk functions (e.g. principal component analysis for fixed-income portfolios) and different quantile estimation methods that can be elaborated on. Furthermore, the multi-horizon model can offer guidance on how to resolve the inherent problem of portfolio insurance models, namely how to reset the floor value. However, more advanced estimators of future returns must be used.

References

ATWOOD, J.A., WATTS, M.J., HELMERS, G.A. (1988). Chance-constrained financing as a response to financial risk. *American Journal of Agricultural Economics* 70(1): 79–89.
http://dx.doi.org/10.2307/1241978

BALDER, S., BRANDL, M., MAHAYNI, A. (2009). Effectiveness of CPPI strategies under discrete time trading. *Journal of Economics and Control* 33: 204–220. http://dx.doi.org/10.1016/j.jedc.2008.04.013

BLACK, F., JONES, R. (1987). Simplifying portfolio insurance. *The Journal of Portfolio Management* 14(4): 48–51.
http://dx.doi.org/10.3905/jpm.1987.409131

BROGAN, A.J., STIDHAM, S. (2005). A note on separation in mean-lower-partial-moment portfolio optimization with fixed and moving targets. *IIE Transactions* 37(10): 901–906. http://dx.doi.org/10.1080/07408170591007803

CHOW, G., KRITZMAN, M. (2001). Risk budgets. *The Journal of Portfolio Management* 27(2): 56–61. http://dx.doi.org/10.3905/jpm.2001.319792

DUC, F., SCHORDERET, Y. (2008). *Market Risk Management for Hedge Funds*. Hoboken NJ: Wiley.

ELTON, E.J., GRUBER, M.J., BROWN, S.J, GOETZMANN, W.N. (2003). *Modern Portfolio Theory and Investment Analysis.* Hoboken, NJ: Wiley.

ENGLE, R.F. (2009). The risk that risk will change. *Journal of Investment Management* 7(4): 24–28.

FABOZZI, F.J., FOCARDI, S.M., KOLM, P.N. (2006). *Financial Modeling of the Equity Market: From CAPM to Cointegration*. Hoboken, NJ: Wiley.

GUIDOLIN, M., TIMMERMANN, A. (2006). Term structure of risk under alternative econometric specifications. *Journal of Econometrics* 131: 285–308. http://dx.doi.org/10.1016/j.jeconom.2005.01.033

HAMIDI, B., JURCZENKO, E., PRIGENT, J.L. (2009). A CAViaR modelling for a simple time-varying proportion portfolio insurance strategy. *Bankers, Markets & Investors* 102: 4–21.

HARLOW, W.V. (1991). Asset allocation in a downside-risk framework. *Financial Analysts Journal* September/October: 28–40. http://dx.doi.org/10.2469/faj.v47.n5.28

HEROLD, U., MAURER, R., PURSCHAKER, N. (2005). Total return fixed-income management. *The Journal of Portfolio Management* 31(3): 32–43. http://dx.doi.org/10.3905/jpm.2005.500351

JIANG, CH., MA, Y., AN, Y. (2009). The effectiveness of the VaR-based portfolio insurance strategy: An empirical analysis. *International Review of Financial Analysis* 18: 185–197. http://dx.doi.org/10.1016/j.irfa.2009.04.001

KAHNEMAN, D., TVERSKY, A. (1992). Advances in prospect theory: cumulative representation of uncertainty. *Journal of Risk and Uncertainty* 5(4): 297–323. http://dx.doi.org/10.1007/BF00122574

LEIBOWITZ, M.L., KOGELMAN, S. (1991). Asset allocation under shortfall constraints. *The Journal of Portfolio Management* 17(2): 18–23. http://dx.doi.org/10.3905/jpm.1991.409309

MEUCCI, A. (2004). Broadening horizons. *Risk* 17(12): 98–101.

MEUCCI, A. (2010). Quant nugget 4: annualization and general projection of skewness, kurtosis and all summary statistics. *GARP's Risk Professional Magazine* July: 59–63.

PEROLD, A.F. (1986). A constant proportion portfolio insurance. *Harvard Business School*, unpublished manuscript.

Additional sources

BROWNLEES, C.T., ENGLE, R.F., KELLY, B.T. (2009). A practical guide to volatility forecasting through calm and storm. *Working Paper*. Available at: ssrn.com/abstract=1502915.

COLACITO, R., ENGLE, R.F. (2009). Term structure of risk, the role of known and unknown risks and non-stationary distributions. *NYU Working Paper No. FIN-08-040*. Available at: http://ssrn.com/abstract =1354527.

GOETZMANN, W.N., BROADIE, M. (1992). Safety first portfolio insurance. *YALE ICF Working Paper* No. 08-11. Available at: http://ssrn.com/abstract=1102328.

PÁSTOR, Ľ., STAMBAUGH, R.F. (2009). Are stocks really less volatile in the long run. *AFA 2010 Atlanta Meetings Paper*. Available at: http://ssrn.com/abstract=1136847.

Appendix A Comparison of two assets: VBPI and CPPI

These portfolio insurance models are mostly suited to dynamic rebalancing methods. Different rebalancing methods that determine the rebalancing period τ_r were highlighted in the article. The analysis of the maximum loss in risky asset value that can be incurred within the rebalancing period keeping a positive value of the cushion is mostly labelled gap risk analysis. Here, we present analytics based on the developed two-asset model and natural extension of CPPI multiplier that express the maximal magnitude of decreases in risky asset value.

The CPPI model calculates the proportion of risky asset w_t from the value of the cushion and multiplier m that controls the leverage of the portfolio:

$$w_t = \frac{m\left(V_t - F\exp(-r\ \tau)\right)}{V_t} \tag{A–1}$$

where F stands for the critical value of the portfolio, called *Floor* in the CPPI model. The proportion of risk-free assets is calculated from the budget constraints. The value of the multiplier can be bounded to constrain short sales or too high leverage and can be defined as a constant or as some function of market variables. In the regular CPPI, the multiplier is set to be the constant and the CPPI model does not require estimation. It can be shown that the inverse of the multiplier equals the maximum considered loss of risky assets that is accommodated by the model. This can be defined in terms of the realisation of a risky asset's linear return as:

$$l_{\tau\_r}^{CPPI} = -m^{-1}. \tag{A–2}$$

If the risky asset fell by more than $l_{\tau r}^{CPPI}$ within the rebalancing period, the critical value requirement is violated and the portfolio value cannot be restored by the risk-free return. The calculation of the corresponding continuously compounded return $c_{\tau r}^{CPPI}$ is straightforward. The natural next step is to express the probability of this boundary return.

The definition of the two-asset VBPI model enables us to create equivalent analytics. There are links between the definitions of the proportion of risky assets in the VBPI model (10) and that in the CPPI model (A–1) where equation (10) can be seen as a more general version of the risky asset proportion calculation. Thus, it is possible to calculate implied multiplier m_{IMP} from the VBPI model:

$$m_{IMP} = \frac{\exp(r\ \tau)}{\exp(r\ \tau) - \exp\left(Q_\alpha\left(c_{T,\tau}^{RA}\right)\right)}. \tag{A–3}$$

The boundary linear and continuously compounded returns induced by the two-asset VBPI model can be derived from equation (A–2) and are equal to:

$$l_{\tau\_r}^{VBPI} = \exp\left(\tau\left(Q(c) - r\right)\right) - 1 \tag{A–4}$$

and

$$c_{\tau\_r}^{VBPI} = \tau\left(Q(c) - r\right), \tag{A–5}$$

where $Q(c)$ is the α-quantile of compounded returns that are expected to be realised at investment horizon T. If the quantile estimation were time-dependent, boundary returns would also be time-dependent. It can be shown that boundary compounded returns can be expressed in terms of the proportions of risky assets and risk-free assets as:

$$c_{\tau\_r}^{VBPI} = \ln\left(\frac{CV - RF_t\exp(r\ \tau)}{RA_t}\right) - r\ \tau, \tag{A–6}$$

where $RF_t = (1 - w_t)V_{t-1}$ and $RA_t = w_t\, V_{t-1}$.

Appendix B Interest rate risk of portfolio insurance models

Risk-free assets are generally assumed to grow linearly over time in portfolio insurance models (see e.g. Balder et al., 2009; Chow and Kritzman, 2001; Jiang et al., 2009). However, the existence of risk-free assets with those characteristics is questionable. Even the Treasury bills of the most credible issuers possess some sort of price volatility and their prices increase rapidly in times of market stress (and yields decrease correspondingly). Thus, in the risk scenario, the risk budget is suppressed because of both a decrease in risky assets and a fall in the risk-free rate. It is possible that the divested risky asset cannot be reinvested at a sufficient risk-free rate. The inverse of the multiplier, therefore, assigns a lower total risk.