Duopoly Pricing Strategy in Spatial Competition Using Constant-sum Games

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***Abstract* – Spatial games focused on imperfect competition are specific area of game theory dealing with behavior of market competitors. This article focuses on the formulation and solution of specific one-round spatial constant-sum game of two players deciding on the location of their branches with aim to attract customers and maximize revenues. The space is characterized by graph, in which nodes represent location of customers and possible place of service. The main goal is to present issue of determining product price of one player, based on predetermined price of his opponent. This determination takes place simultaneously with the location selection of both players.**

***Keywords* – duopoly, imperfect competition, matrix games, pricing, spatial competition.**

1. Introduction

The basis of an open market economy is free competition, which is a conflict of interest of market players. To operate in the market and maintain or improve its position, company has to be competitive and proceed strategically.

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The concept of strategy can be found in various fields, one of which is game theory, a tool for analyzing the strategic behavior of players who can represent any entity in a conflict decision-making situation. In the market, it is precisely companies that find themselves in a conflict situation with competing companies offering the same or similar products and services. Each of these companies aims to gain as many customers as possible, increase their market share and maximize their profits. How a company behaves in the market is also influenced by the type of market structure in which it operates. In the market, we generally distinguish between perfect and imperfect competition, both of which are characterized by their specific features [2]. The behavior of the firms in the market of imperfect competition has to consider the decisions and steps of other subjects (other companies). The path to success is, except for market structure, defined by several factors, like choice of location that can be defined at the level of municipalities, cities, regions, or states. Closely related to this is spatial competition, which transfers the competitive environment to space. The space and dynamics of relations between subjects lead to the emergence of the mentioned spatial competition. In the models focused on this topic, companies compete mainly in prices and their locations. Together, they form the basic factors that determine their market presence.

Many authors in the world deal with the application of spatial economics and spatial competition, while their approach is different. The combination of spatial competition with the game theory is a specific and very promising topic due to market dynamics and development of strategic business management methods. The aim of this article is to point out the connection between game theory and spatial competition, specifically to approach the possibilities of using one-round constant-sum games in spatial competition, from the point of view of duopolist pricing policy. Given that authors who have dealt with this issue in the past have presented models based on sequential methods, our paper should provide a different perspective about decision-making on locations and prices, as we do not assume a decision in stages, but demonstrate a situation where players decide simultaneously. We assume a situation where two companies enter the market at the same time and both decide on the location of the place of service, but one of them has a priori known information about the price of a competitor. So, one player chooses his placement and the other chooses both location and his price.

We present the duopolistic market in a space that can be characterized in the form of a graph [2]. It is one-round game, the results of which are determined by setting the prices of products of the companies, which also affect their respective market share (and thus sales). At the same time, customers choose one of the two duopolists based on their lower costs, which include both the price of the goods and the transport costs. We will also assume that the product price of one of the duopolists is known in advance. We will design an original mathematical model, based on which it is possible to set the product price for the opponent so that his sales are as high as possible.

The article is divided into following interrelated parts. In the first section we offer a literature review of works focused on spatial competition. In the second section we will present a specific spatial game, where the goal of duopolists is to maximize their market share (and related revenues), determined by the number of nodes served. We assume that each customer has to be served by one of the duopolists. The situation can then be formulated as constant-sum game. We will also present an original mathematical model for determining the product price of one of the duopolists, which is the best response to the known product price of the opponent. The use of the model we present on an illustrative example. In the third section we modify this model by introducing different demand of the customers and subsequently limited capacity and offer of duopolists. We also illustrate this case with an example. The last section is devoted to illustrative example inspired by the administrative division of the Slovak Republic, using the presented approaches.

The problems presented in the article are solved by GAMS (General Algebraic Modeling System). It is one of the software tools for solving more complex optimization problems and focuses mainly on modelling linear, nonlinear and mixed integer problems. After initial experiments, we chose to use the Couenne solver (version 25.1.3), because its use has proved to be the most advantageous in terms of time and quality of the achieved results. Couenne (Convex Over and Under ENvelopes for Nonlinear Estimation) is a new branch and boundary algorithm like the Baron or LindoGlobal solvers, designed to solve a class of non-convex mixed integer programming (MINLP) problems. More information about the individual steps of the solution can be found in the official manual by [1]. All experiments were performed on a PC with an Intel® Core® i7-8650U CPU @ 1.9 GHz 2.10 GHz with 16 GB RAM. The calculations of the presented illustrative examples were performed in the range of 5s -20 min.

2. Models of Spatial Competition

The analysis of the oligopolistic market in space is currently increasingly discussed topic [2]. One of the first to address this issue was the mathematician and economist H. Hotelling [3], who presented a model based on the presence of two companies looking for the most advantageous position in the linear market. The model is the basis of many theories of product differentiation and location. Summary of the Hotelling’s work, his methodology, concepts and contribution from a mathematical point of view, in order to better understand its models, can be found in [4]. Despite applicability of Hotelling’s model, it has undergone many criticisms. For example, [5] point out its flaw and proves that it is not possible to have equilibrium if companies are close to each other. The result of their modified model is a model whose solution ensures the existence of equilibrium at any point in the market. This model was extended by [6] to continuous time model. Authors focus mostly on the decisions of timing of entering the market while choosing the locations.

Even though the beginnings of the issue of spatial models are associated with Hotelling, in fact, there were few authors analyzing economic activities in space before him. In 1924 Fetter [7] published work with a significant impact on network competition theory. Unlike Hotelling, he focused on modelling demand behavior, not on optimal decisions [8]. A further extension of Fetter’s work can be found in the publications of many other authors, such as [9], [10]. In [9] authors focused on the analysis of price equilibrium in a two-stage game of the model of spatial competition of a linear city. In [11] authors consider a circular, not a linear market. Other publications following Hotelling’s model are [12] or [13] where authors deal with the price competition of the spatial duopoly. Customers located along the linear market, forced to travel if they want to buy the products on offer, are the only ones who bear the transport costs. The location is an exogenous parameter for the companies, so price is their only decision variable [14]. The fact that the Hotelling’s model has laid the foundations for this issue also proves its applicability in other areas, such as political science. Use of its extension, Hotelling-Downs model, used for analysis of political competition and strategies is presented for example in [14] or [15]. Other extension, Hotelling-Smithies game (Hotelling game with elastic demand) is used in [16] to analyze correctness of game solution in case of games with variable location and fixed or variable price. In such games the simulations lead to the same conclusions as Hotelling in his work (locations tend to drift towards the center).

There are also more recent publications following the basics of the spatial competition models, developing the topic using more conditions. [17] propose a new two-stages position model with one leader choosing location and one or more followers trying to choose their location as close to it as possible. Available spatial and demographic information affect not only business location decisions, but also land use planning decisions. In [18] authors model a game in which players are represented by landowners in order to use it as efficiently as possible. Such available information is used in their analysis by Olszewski et al. [19]. They focus on the use of different types of public policy in order to find the most profitable one for a city and its self-government, but also its businesses and citizens. [20] propose contribution of spatial competition to the Industrial Revolution and the Great Divergence and based on relations between the spatial competition, innovation and craft guilds claim that industrialization depends not only on the market size, but the competition between the craft guilds. Authors in [21] investigated a zonal mechanism used to find the location of two companies in linear space. Regulator enters their model in three ways. In the first case, it prevents companies from locating their branches outside the city limits. In the second case, the regulator is interested in companies and allows the location of branches outside the city limits within the extended zone. In the third case, the regulator is biased against companies and allows them to be located only outside the city limits within a certain distance. National supervision and regulation are also subject of [22], where it focuses on the safe production of coal mines and regulatory supervision, conditioned by government support. In [23] an analysis of the strategies of national governments, local governments and enterprises in China can be found, to examine environmental regulation policies.

The topic of competition and its extension into online platforms, mostly nowadays when its popularity is growing. Analysis of impact of spatial differentiation on pricing mechanism of online car hailing platform is presented in [24]. Also [25] point to growth of e-commerce of grocery retail and importance of incorporating it into the location models. Use of Hotelling's model is presented in [26] for analysis of customization strategy of offline retailer to make him competitive to online retailer with the same product.

[8] distinguish a total of four types of research paths that were formed after Hotelling's work, according to the number and frequency of publications that were created, the approach to spatial competition and the approach to corporate strategies. The first group is Bertrand's competition, which is directly linked to Hotelling, and its solution represents equilibrium where companies choose the prices of their production, the second is Cournot's competition focusing on the quantities of production. The third group are models with nonlinear markets (circular, triangular) and the last fourth are models of incomplete information extending models with complete information of players. However, Bertrand's and Cournot's equilibria are the best known and most common of them. We can say that most often used are iterative models in which companies choose first the location of their branches and then the prices of their products (or vice versa).

The second most common, Cournot's equilibrium, is also considered as a prototype of Nash's equilibrium in non-cooperative decision-making situations. However, the existence of Nash equilibrium is very discussed in the analysis of such two-stage games, in which the location of the company is chosen first and then the price (Bertrand) or quantity (Cournot) of production. When analyzing the existence of equilibrium, it is important to distinguish whether it is a simultaneous or sequential game and whether players decide only on their location or price. The first case, when companies simultaneously choose their positions, corresponds to the Cournot-Nash equilibrium when none of the players wins by one-sided deviation from the solution obtained (such locations should be close to each other in the center of the market). If we also add prices as variables to such model, this concept of equilibrium no longer applies. In such situation, it is more sensible for companies to choose their position first (as it is more difficult to change it) and only at later stage their prices [27]. The authors further explain the solution of such game as follows: the game is solved recursively, i.e., players determine their pricing functions with respect to all possible localization arrangements. The Cournot-Nash equilibrium is obtained if players maximize their profits. Equilibrium payoffs then come into game for placement in the first stage, for which the Cournot-Nash equilibrium is determined as in the first case - the equilibrium position. This solution is known as the subgame-perfect Nash equilibrium developed by [28]. [29] argues that for existence of equilibrium, location and price cannot be variables, but prices have to be set in advance. [30] introduced the concept of solution in mixed strategies into spatial models where there is no equilibrium in pure strategies, which further elaborates [31] or [32]. This brings a new approach to finding equilibrium in spatial models.

3. Determining the Product Price of Duopolist Based on the Best Response

In this section, we will present an original mathematical model that allows us to determine the price of the product of duopolist based on the determined price of the opponent in the case of a specific spatial game. In the model, like Hotelling in his basic model, we apply the basic assumptions: product homogeneity (both companies on the market offer a very similar product), zero production costs of companies, inelasticity of demand (consumption of one unit by customers at each market point) and consumer indifference (they choose the producer without preference, only according to their total costs). Every customer wants to be served, which means that the goal is to serve the entire market, while the aim of each producer is to maximize his revenue [2]. Although most of the literature deals with sequential models, in which companies first choose their location and in the second step the prices of their products (or vice versa), we bring model representing static (one-round) game, in which the choices of price of one player and locations of both players take place simultaneously in the same step.

We will assume following [2]: Let $V = \{1,2,\dots n\}, n \in Z^+$ be the set of customers and let there be graph $G = (V,H)$ where V represents nodes of the graph and $H \subset VxV$ represents set of the edges $h_{ij} = (i,j)$ from node i to node j, while for each oriented edge h_{ij} there is assigned real number $o(h_{ij})$ referred to as a valuation or value h_{ij}. Spatial game was formulated in so-called full-valued graph $\bar{G} = (V, \bar{H})$ with the same set of nodes as graph G, where $\bar{H}$ is set of the edges between each pair of nodes i and j, while their valuation is equal to the minimum valuation between nodes i and j of the original graph. It is often assumed that $o(h_{ij}) = d_{ij}$ where d_{ij} represents the minimum distance (the shortest path lenght) between the nodes i and j, then the matrix $\mathbf{D}_{nxn} = \{d_{ij}\}$ is the matrix of the shortest distances between the nodes i and j.

We assume two companies (players) $P = \{1,2\}$, offering a homogeneous product (good or service) in unlimited quantities, and these companies can place their branches in just one of the nodes, i.e., in any element of the set V, which are also the locations of customers. If such solution does not exist, we assume the possibility of dividing the service, while considering the sources of service of both players to be the same size. We also consider a constant (unit) demand at each node. Although both players offer identical products in unlimited quantities, the price of the products may be different. Let p_1 be the price of the product of player 1 and p_2 the price of the product of player 2. Each customer makes a purchase from any company (service is always carried out, i.e., lost demand is not considered).

The total cost of purchasing the product by customer consists not only of its price, but also of the cost of transportation to the selected company and we assume him to prefer the lower one. Transport costs are expressed as t per unit distance. Let the total costs of customer in ith node linked to the service of player in jth node be represented by coefficients $t * d_{ij} + p_1 = n_{ij}^{(1)}$ for player 1 and $t * d_{ij} + p_2 = n_{ij}^{(2)}$ for player 2 (the coefficients can be written in cost matrices of customers $\mathbf{N}^{(1)}$ and $\mathbf{N}^{(2)}$). The indifference of customers is represented by parameter ε, which can be considered as small positive number. That means that the customer is indifferent in choosing the company if their total costs do not differ more than $\pm\varepsilon$. Bertrand in his duopoly model (1883) also considered this parameter as the smallest possible positive number in setting price of one player's price and stated that slight reduction of price of this player will lead to double sales and very small decline of his profit margin per unit sold ($p_2^* = p_1 - \varepsilon$).

Let us suppose that player 1 places his store in the ith node and player 2 places his store in the jth node, player 1 gets the customer of kth node ($i,j,k \in V$) if $n_{kj}^{(2)} - n_{ki}^{(1)} \geq \varepsilon$, in case that $n_{ki}^{(1)} - n_{kj}^{(2)} \geq \varepsilon$ he gets demand of node of player 2. If $abs\left(n_{kj}^{(2)} - n_{ki}^{(1)}\right) < \varepsilon$ players get half of the node's demand.

The basic situation, represented by a fixed price model [2], is where the prices of both products are known in advance and based on the above assumptions. Thus, elements of the payment matrix of player 1 $\mathbf{A} = (a_{ij}), i,j \in V$, (where the element a_{ij} represents the number of served nodes of player 1 in the case if player 1 operates in the ith node and the opponent in the jth node), are explicitly calculated. The elements of matrix $\mathbf{A}$ are quantified based on the stated elements of cost matrices of consumers as follows:

LET $V = \{1,2,\dots n\}$, $\mathbf{D}_{nxn} = \{d_{ij}\}$, t, $p^{(1)}$, $p^{(2)}$
LOOP $(i,j \in V)$ **DO**
$n_{ij}^{(1)} = t * d_{ij} + p^{(1)}$;
$n_{ij}^{(2)} = t * d_{ij} + p^{(2)}$;
$a_{ij} = 0$;
LOOP $(k,i,j = 1,2,\dots n)$ **DO**
IF $n_{kj}^{(2)} - n_{ki}^{(1)} \geq \varepsilon$ **DO** $a_{ij} = a_{ij} + 1$;
ELSEIF $abs\left(n_{kj}^{(2)} - n_{ki}^{(1)}\right)$ **DO** $a_{ij} = a_{ij} + 0.5$;
ENDIF

Such matrix characterizes given game with a constant sum (where the game constant is equal to the number of nodes of the graph G). Equilibrium strategies can then be determined in a standard way based on the min-max principle. If the use of this approach does not lead to an equilibrium strategy, equilibrium strategies can be determined based on linear programming problem (that means we accept equilibrium solution in mixed strategies if there is no Nash equilibrium in pure strategies).

The game is in this case given by these parameters:

- $a_{ij}, i, j \in V$ – payment of the player at his ith strategy and jth strategy of his opponent
- and variables:
- w – final payment of the player (number of served nodes)
- $x_i \in \langle 0,1 \rangle$, $i \in V$ – ith mixed strategy of the player

The equilibrium strategies can be determined as follows:

$$w \to \max \tag{1}$$

$$\sum_{i \in V} a_{ij} x_i \geq w, j \in V \tag{2}$$

$$\sum_{i \in V} x_i = 1 \tag{3}$$

Payment matrix of player 2 ($\mathbf{B} = (b_{ji}), i, j \in V$) can be determined as $\mathbf{B} = n - \mathbf{A}^T$, where n represents number of the nodes. Strategies of player 2 ($y_j \geq 0, j \in V$) can be determined as dual variable belonging to the equations (2). Value of the game of player 2 (u) can be calculated as $u = n - w$.

In the previous text we assumed that the prices of both players are known in advance and that the players decide only on their placement. Let us now assume a situation where one of the players has information about the price of his competitor in advance and thus can decide not only on his location, but also on the price, while he wants to set it in order to maximize his revenues.

Now let us look at how to determine the best response to an opponent's price in the following example. We consider the set $V = \{1,2,3,4\}$, which represents closed set of customers. Each of these customers are located in one of the four nodes. We consider the unit cost of delivery of goods in the amount of $t = 1$. Let the shortest distances between the nodes represent the elements of the matrix $\mathbf{D} = d(i,j)$:

$$\mathbf{D} = \begin{bmatrix} 0 & 6 & 7 & 9 \\ 6 & 0 & 12 & 15 \\ 7 & 12 & 0 & 11 \\ 9 & 15 & 11 & 0 \end{bmatrix}$$

The price of the goods of the second player p_2 is known in advance. Let $p_2 = 1$. We assume the customer's tolerance to be $\varepsilon = 0.001$.

At first, we will use fixed price model, where we will consider changing price of the goods of the first player p_1 in interval $\langle \varepsilon, 25 \rangle$ by step size ε. The player would like to set it in a way to maximize his revenues.

Final values of the game for player 1, depending on his value set to p_1 are presented in the Figure 1.

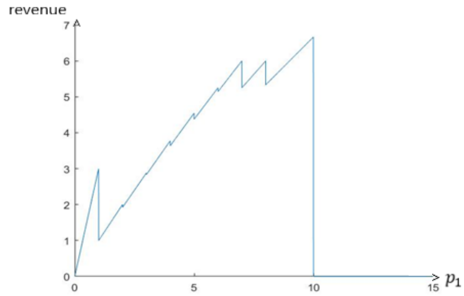

Figure 1. Revenue values obtained using fixed priced model

The resulting value of the game for player 1 is given by piecewise linear function, the threshold of which are related to the fact whether the player "gains" or does not gain service of a node, while player 1 is located in ith and player 2 in jth node. The fact whether the player gains the node depends on the total costs of customers.

For the above case, the thresholds are p_1 ={1-ε,1+ε ,2-ε, 2+ε,3-ε, 3+ε, 4-ε, 4+ε,5-ε, 5+ε, 6-ε, 6+ε, 7-ε, 7+ε, 8-ε, 8+ε ,10-ε, 10+ε,12-ε, 12+ε,13-ε,13+ε, 6-ε, $6 + \varepsilon$}. The way of solution is therefore to calculate revenues for each of these values and then to select the one, at which revenues reach the maximum. In this case the maximum revenue of 6.666 the player will get with the price of his product at level $p_1^* = 9.999$, what can also be seen in the Figure 1 (even though for price $p_1^* = p_2 - \varepsilon = 1 - 0.001 = 0.999$, the player gets 2 nodes, but his revenue will be only 1.998).

Let us now look at the problem from a different perspective. The aim is to design a mathematical model that will allow us to obtain the above results in one step. It is obvious that under such assumptions the elements of the payment matrix of player 1 will depend on the value of p_1. At the same time both players also make decisions about their locations. If player 1 decides to build his place of consumption in the ith node and player 2 in the jth node, then for the customer from kth node, while choosing the player, three cases can occur:

a) $t * d_{kj} + p_2 - (t * d_{ki} + p_1) > 0$

b) $t * d_{kj} + p_2 - (t * d_{ki} + p_1) = 0$

c) $t * d_{kj} + p_2 - (t * d_{ki} + p_1) < 0$

Assigning a customer to player 1 can then be done using the Signum function:

$$sgn(t * d_{kj} + p_2 - (t * d_{ki} + p_1))$$

where case a) returns value of 1, case b) returns value of 0 and case c) returns value of -1. As in the case a) the *k*th node will be assigned to player 1 (demand of 1), in the case b) the half of the *k*th node's demand will be assigned to player 1 (demand of ½) and in the case c) player 1 will not serve *k*th node. Relationship between the elements of the payment matrix and the price p_1 is:

$$a_{ij}(p_1) = \sum_{k \in V} \frac{sgn\left(t * d_{kj} + p_2 - (t * d_{ki} + p_1)\right) + 1}{2}, i, j \in V$$

Now it is possible to express the pricing for player 1 by this mathematical model:

$$w * p_1 \to \max$$

$$a_{ij}(p_1) = \sum_{k \in V} \frac{sgn\left(t * d_{kj} + p_2 - (t * d_{ki} + p_1)\right) + 1}{2}, i, j \in V$$

$$\sum_{i \in V} a_{ij} x_i \geq w, j \in V$$

$$\sum_{i \in V} x_i = 1$$

The problem is discontinuity of the Signum function here. Now we will present how to replace this function by binary programming problem. Let us introduce binary variables $b_{kij}^{(1)} \in \{0,1\}$ and $b_{kij}^{(2)} \in \{0,1\}; k,$ and continuous variable $b_{kij} \in \langle -1,1 \rangle;\ k, i, j \in V$ and add following equations:

$$t * d_{kj} + p_2 - (t * d_{ki} + p_1) \leq M * b_{kij}^{(1)}; k, i, j \in V \quad (4)$$

$$t * d_{kj} + p_2 - (t * d_{ki} + p_1) \geq -M * b_{kij}^{(2)}; k, i, j \in V \quad (5)$$

where M is big positive number. If the player's price upper limit is known, it can be considered (depending on the character of the problem) as:

$$M > t * \max_{i,j \in V}(d_{ij}) + (p_1^{up} - p_2)$$

while p_1^{up} is the upper limit of the product price of player 1 and $\max(d_{ij})$ is the maximum shortest distance between node *i* and node *j*.

By the equations (4) we ensure that if $t * d_{kj} + p_2 - (t * d_{ki} + p_1) > 0$, then $b_{kij}^{(1)} = 1$ and by the equations (5) we ensure that if $t * d_{kj} + p_2 - (t * d_{ki} + p_1) < 0$, then $b_{kij}^{(2)} = 1$. Because these two cases cannot occur simultaneously, we will introduce equations:

$$b_{kij}^{(1)} + b_{kij}^{(2)} \leq 1; k, i, j \in V$$

Calculation of elements a_{ij} we will ensure using variables $b_{kij} \in \langle -1,1 \rangle;\ k, i, j \in V$:

$$b_{kij} = b_{kij}^{(1)} - b_{kij}^{(2)}; k, i, j \in V \quad (6)$$

$$b_{kij}^{(1)} * (t * d_{kj} + p_2 - (t * d_{ki} + p_1)) \geq \varepsilon * b_{kij}^{(1)}; k, i, j \in V \quad (7)$$

$$b_{kij}^{(2)} * \left(t * d_{kj} + p_2 - (t * d_{ki} + p_1)\right) \leq -\varepsilon * b_{kij}^{(2)}; k, i, j \in V \quad (8)$$

$$a_{ij} = \sum_{k \in V} (b_{kij} + 1)/2; i, j \in V \quad (9)$$

where equations (7) and (8) deal with case, when $t * d_{kj} + p_2 - (t * d_{ki} + p_1) = 0$.

Let us summarize the basic Best response model given by these sets and parameters:

- $n \in Z^+$ – number of nodes
- $V = \{1,2, \dots n\}$ – set of all nodes
- $d_{ij} \geq 0, i, j \in V$ – shortest distance between nodes *i* and *j*
- $t > 0$ – costs per unit distance
- $p_2 > 0$ – price of product of player 2
- M – big positive number
- $\varepsilon -$ small positive number

and variables:

- $w \in \langle 0, n \rangle$ – number of served nodes
- $x_i \in \langle 0,1 \rangle$, $i \in V$ – *i*th mixed strategy of player 1
- $p_1 > 0$ – price of product of player 1
- $a_{ij} \in \langle 0, n \rangle, i, j \in V$ – elements of payment matrix of player 1
- $b_{kij}^{(1)} \in \{0,1\}; k, i, j \in V,$
- $b_{kij}^{(2)} \in \{0,1\}; k, i, j \in V,$
- $b_{kij} \in \langle -1,1 \rangle;\ k, i, j \in$ V.

The final form of the model is:

$$w * p_1 \to \max \quad (10)$$

$$t * d_{kj} + p_2 - (t * d_{ki} + p_1) \leq M * b_{kij}^{(1)}; k, i, j \in V \quad (11)$$

$$t * d_{kj} + p_2 - (t * d_{ki} + p_1) \geq -M * b_{kij}^{(2)}; k, i, j \in V \quad (12)$$

$$b_{kij}^{(1)} + b_{kij}^{(2)} \leq 1; k, i, j \in V \quad (13)$$

$$b_{kij} = b_{kij}^{(1)} - b_{kij}^{(2)}; k, i, j \in V \quad (14)$$

$$b_{kij}^{(1)} * (t * d_{kj} + p_2 - (t * d_{ki} + p_1)) \geq \varepsilon * b_{kij}^{(1)}; k, i, j \in V \quad (15)$$

$$b_{kij}^{(2)} * \left(t * d_{kj} + p^{(2)} - \left(t * d_{ki} + p^{(1)}\right)\right) \leq -\varepsilon * b_{kij}^{(2)}; k, i, j \in V \quad (16)$$

$$a_{ij} = \frac{\sum_{k \in V}(b_{kij} + 1)}{2}; i, j \in V \quad (17)$$

$$w \leq \sum_{i \in V} a_{ij} * x_i\, ; j \in V \quad (18)$$

$$\sum_{i \in V} x_i = 1 \qquad (19)$$

The objective function (10) represents the revenue function of player 1. Equations (11) to (17) are used to determine the payment matrix of player 1. Equations (18) and (19) make it possible to determine the equilibrium mixed strategy of player 1.

If we also want to know the mixed strategy of the opponent, following conditions can be added to the model:

$$w \geq \sum_{i \in V} a_{ij} * y_j \,; j \in V \qquad (20)$$

$$\sum_{j \in V} y_j = 1 \qquad (21)$$

where $y_j \geq 0, j \in V$, is jth element of an opponent's mixed strategy.

Generally, the mathematical model can find equilibrium strategy in mixed strategies. Although this is not unrealistic (with the divisibility of resources), but in several practical problems it is necessary to implement a clear decision (pure strategy).

Let us now consider that although player can divide his resources among several nodes, every service point is associated with additional costs at the same time, so it is more advantageous to realize the service in as few places as possible (ideally to pursue the pure strategy). Let the costs be the same for each node and their amount is determined by the constant h. Let us introduce new binary variables $bin_i^{(1)} \in \{0,1\}; i \in V$ into the model. The new variables will represent realisation of service in ith node (if the service takes place in the ith node, $bin_i^{(1)} = 1$, otherwise $bin_i^{(1)} = 0$). We can add following constraints into the model:

$$x_i \leq bin_i^{(1)}, i \in V \qquad (22)$$

while changing the objective function:

$$w * p_1 - h * \sum_{i \in V} bin_i^{(1)} \to \max \qquad (23)$$

If we want to get a solution in the pure strategies also for player 2, we need to add new binary variables $bin_j^{(2)} \in \{0,1\}; j \in V$ into the model a add the constraints:

$$y_j \leq bin_j^{(2)}, j \in V \qquad (24)$$

$$\sum_{j \in V} bin_j^{(2)} = 1 \qquad (25)$$

It is obvious that under the given conditions there is always at least one solution in the pure strategies for player 1, such that players perform their service in the same node ($bin_i^{(1)} = bin_i^{(2)} = 1$), for the same price ($p_1 = p_2$), sharing the total demand so that $w = \frac{n}{2} - t$ (this value can also be considered as the lower limit of the objective function).

3.1. Numerical Example

Following the initial data from the previous example and based on our Best response model given by equations (10) - (19), we obtain the results, using GAMS and its Couenne solver (this is a problem of mixed integer nonlinear programming (MINLP)). The calculated price of the first player's product at the known price $p_2 = 1$ is at level $p_1^* = 9.999$. The solution also gives the final payment matrix **A**. The payment matrix **B** can be determined as $\mathbf{B} = n - \mathbf{A}^T$.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 4 & 4 & 4 & 3 \\ 4 & 4 & 3 & 3 \\ 4 & 3 & 4 & 3 \\ 2 & 3 & 3 & 4 \end{bmatrix}$$

In this case we will present also the results obtained by Fixed price model, which can be used to verification of results obtained using the Best response model. We write the elements $n_{ij}^{(1)}$ and $n_{ij}^{(2)}$ in the cost matrices (that correspond to the payment matrices) $\mathbf{N}^{(1)} = \left(n_{ij}^{(1)}\right)$ and $\mathbf{N}^{(2)} = \left(n_{ij}^{(2)}\right)$:

$$\mathbf{N}^{(1)} = \begin{bmatrix} 9.999 & 15.999 & 16.999 & 18.999 \\ 15.999 & 9.999 & 21.999 & 24.999 \\ 16.999 & 21.999 & 9.999 & 20.999 \\ 18.999 & 24.999 & 20.999 & 9.999 \end{bmatrix}$$

$$\mathbf{N}^{(2)} = \begin{bmatrix} 1 & 7 & 8 & 10 \\ 7 & 1 & 13 & 16 \\ 8 & 13 & 1 & 12 \\ 10 & 16 & 12 & 1 \end{bmatrix}$$

The equilibrium strategy of the first player is given by vector $\mathbf{x}^T = (0.333; 0; 0; 0.667)$. With the price $p_1^* = 9.999$, player 1 will serve 0.666 nodes ($w = 0.666$) on average. His final average revenue will then be 6.666. The strategy of second player is given by vector $\mathbf{y}^T = (0; 0.333; 0.333; 0.333)$.

Interpretation of mixed strategies (the probability of strategy selection) is generally difficult. If it was possible to change the place of service (for example, daily revenues or distribution of the number of employees), player 1 should perform 66.7% of service in node 4 and 33.3% in node 1. Player 2 should perform 33.3% of service evenly in nodes 2, 3 and 4. Let us now suppose costs associated with the location of the service for player 1. Let such costs be $h = 4$. Under this condition, with the two components of mixed strategy (in the nodes 1 and 4) would the player's revenue drop to -2.666 ($6.666 - 2 * 4$).

Let us now introduce conditions (22) and objective function (23) into the model used in previous example. In this case we obtain new payment matrix:

$$\mathbf{A} = \begin{bmatrix} 4 & 3 & 3 & 3 \\ 1 & 4 & 2 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 2 & 1 & 4 \end{bmatrix}$$

The equilibrium solution for player 1 is $x_1 = 1$ with revenue of -1,003. Player 1 sets price of his product to $p_1^* = 0.999$ (just below the price of player 2) and gets demand of 3 nodes ($w = 3$). However, this solution does not represent the saddle point of the matrix. If we want to get solution in the pure strategies (we suppose that player 2 takes a clear position and perform the service only in one node), we can add conditions (24) and (25) into the model. The new solution will be $x_1 = 1$ and $y_1 = 1$, which corresponds with the saddle point of the matrix **A** (marked):

$$\mathbf{A} = \begin{bmatrix} \mathbf{2} & 3 & 3 & 3 \\ 1 & 2 & 2 & 2 \\ 1 & 2 & 2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

Value of revenue of player 1 will be then -2 at price equal to the price of player 2 (the most advantageous for the player is to divide the market in half in this case).

In the next section, we will extend this model by including assumptions of limited capacity and offer of player.

4. Determining the Product Price of Duopolist Considering His limited Capacity and Different Demands of Nodes

In the previous section we considered unit demand of individual nodes of the graph. It is obvious that a player's interest in each node is generally conditioned by the "size" of the demand of a given node, while in terms of this criterion; some nodes are more interesting for the player than the others. The size of demand can be related, for example, to the number of inhabitants. We will also leave the assumption of an unlimited offer and assume the limited offer of players. Considering the limited demand of nodes, which is given by the vector $\mathbf{g} = (g_i), i \in V$, let us also consider constraints on supply side. We will mark the maximum offered quantity of goods for player 1 as k_1 and k_2 for player 2. Consumer demand will then be distributed among the players based on the following rules: the consumer seeks to minimize his costs. However, if the player's capacity is not sufficient, he has to, despite the increased costs, move to the opponent [2].

When solving such game, the total capacity on the supply side needs to be considered. If it is possible to satisfy the whole demand of the nodes, that means if $k_1 + k_2 \geq \sum_{i \in V} g_i$, it is possible to formulate it as a game with constant sum. If it is not possible to satisfy the whole demand of the nodes, that means if $k_1 + k_2 < \sum_{i \in V} g_i$, any node would be equally advantageous for both players ($\mathbf{A}_{nxn} = (k_1), \mathbf{B}_{nxn} = (k_2)$). In case p_1^{up} is known, the solution is obvious and set to the limit p_1^{up}.

In case the prices of duopolists are known in advance (case of the Fixed price model) and it is possible to satisfy the whole demand of the nodes ($k_1 + k_2 \geq \sum_{i \in V} g_i$), the calculation of elements of payment matrix of player 1 (**A**) can be written in the form of the following procedure [2]:

LET $V = \{1,2, \dots n\}, \mathbf{D}_{nxn} = (d_{ij}), t, p_1, p_2, \mathbf{g}_n = (g_i), k_1, k_2$
LOOP $(i, j \in V)$ **DO**
$n_{ij}^{(1)} = t * d_{ij} + p_1;$
$n_{ij}^{(2)} = t * d_{ij} + p_2;$
$a_{ij} = 0;$
LOOP $(k, i, j \in V)$ **DO**
IF $n_{kj}^{(2)} - n_{ki}^{(1)} \geq \varepsilon$ **DO** $a_{ij} = a_{ij} + g_k;$
ELSE $abs(n_{kj}^{(2)} - n_{ki}^{(1)})$ **DO** $a_{ij} = a_{ij} + 0.5 g_k;$
ENDIF
LOOP $(i, j \in V)$ **DO**
IF $a_{ij} - k_1 > 0$ **DO** $a_{ij} = k_1;$
ENDIF

Let us now show how to determine the equilibrium price of player as the best response to the set price of the opponent, but with different size of the demand of individual nodes represented by vector $\mathbf{g} = (g_i), i \in V$ and the equation (17) will be replaced by equation:

$$a_{ij} \leq \frac{\sum_{k \in V}(b_{kij} + 1)}{2} * g_k; i, j \in V \tag{26}$$

Now we consider the case where duopolist knows the limit of his capacity, but he does not know capacity of his opponent (he supposes it to be large enough to satisfy the whole demand).

$$a_{ij} \leq k_1; i, j \in V \tag{27}$$

These equations will ensure the setting of such values of matrix **A**, that also meet the capacity limit for player 1.

4.1. Numerical Example

Let us follow up on the previous example. The demand of individual nodes is given by the vector $\mathbf{g} = (10; 10; 30; 10)^T$ and $p_2 = 1$. Based on the equations (10)-(16), (18), (19) and (26), an equilibrium price for player 1 $p_1^* = 6.999$ can be obtained. At such price, player 1 serves 12 units of demand (w), with revenue of 83.988. Player 2 serves 48 units of demand (60-12). In this case we will state both payment matrices **A** and **B** ($\mathbf{B} = \sum_{i \in V} g_i - \mathbf{A}^T$), which are as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 20 & 20 & 20 \\ 10 & 0 & 10 & 10 \\ 30 & 30 & 0 & 30 \\ 10 & 10 & 10 & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 60 & 50 & 30 & 50 \\ 40 & 60 & 30 & 50 \\ 40 & 50 & 60 & 50 \\ 40 & 50 & 30 & 60 \end{bmatrix}$$

Equilibrium strategies of players are represented by vectors $\mathbf{x}^T = (0.6; 0; 0.4; 0)$ and $\mathbf{y}^T = (0.4; 0; 0.6; 0)$.

Let us now consider situation, where the costs associated with the location of the service for player 1 are $h = 4$. Using model (11)-(16), (18), (19), (22), (23), (26) we obtain following solution:

The matrix **A** will be:

$$\mathbf{A} = \begin{bmatrix} 60 & 50 & 30 & 50 \\ 10 & 60 & 20 & 20 \\ 30 & 40 & 60 & 50 \\ 10 & 40 & 10 & 60 \end{bmatrix}$$

and equilibrium solution is given by vectors $\mathbf{x}^T = (0.5; 0; 0.5; 0)$ a $\mathbf{y}^T = (0.071; 0.643; 0.286; 0)$. Found price of the product of player 1 at known price of product of player 2 $p_2 = 1$ is just below its level at $p_1^* = 0.999$. At this price, the player serves a total of 45 units of demand ($w = 45$) and achieves a revenue of 36.955.

Let us now extend this example with the capacity constraint for player 1. Let $k_1 = 10$. Capacity of second player is large enough to satisfy the rest of the market demand. Using model (10)-(16), (18), (19), (26) and (27) will change the solution as follows:

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 5 & 10 \\ 0 & 0 & 10 & 10 \\ 10 & 10 & 0 & 10 \\ 10 & 10 & 10 & 0 \end{bmatrix}$$

Equilibrium strategies of players are $\mathbf{x}^T = (0; 0.333; 0.333; 0.333)$ and $\mathbf{y}^T = (0.295; 0.039; 0.333; 0.333)$. Equilibrium price of player is $p_1^* = 8$. At such price player 1 will serve 6.667 units of demand (w) and his revenue will be 53.333.

Let us now introduce operating costs of $h = 4$ into the model again. Given the assumptions and using model (11)-(16), (18), (19), (22), (23), (26) and (27) we get payment matrix **A**:

$$\mathbf{A} = \begin{bmatrix} \mathbf{10} & \mathbf{10} & \mathbf{10} & \mathbf{10} \\ \mathbf{10} & \mathbf{10} & \mathbf{10} & \mathbf{10} \\ \mathbf{10} & \mathbf{10} & \mathbf{10} & \mathbf{10} \\ \mathbf{10} & \mathbf{10} & \mathbf{10} & \mathbf{10} \end{bmatrix}$$

The found price of player 1 will be the same as pre-known price of player 2, i.e., $p_1^* = 1$. With such price he will serve 10 units of demand (w) and achieve revenue of 6. The solution is known as weak equilibrium as we get more saddle points (all 16 points marked in this case).

We will use the new models presented above in an illustrative example in the next section.

5. Location of the Service Point in One of the Regional Cities of SR

Further illustrative example is inspired by the administrative division of the Slovak Republic [2]. Let the nodes of graph *G* represent potential regions for the construction of new branches of two companies operating in the market in the position of two strong players (P = {1,2}). By regions we will understand the regions of the Slovak Republic, represented by individual regional cities: 1-Banská Bystrica, 2-Bratislava, 3-Košice, 4-Nitra, 5-Prešov, 6-Trenčín, 7-Trnava and 8- Žilina. These 8 cities therefore represent the nodes of the graph *G*, $V = \{1,2, \dots 8\}$.

The demand of individual nodes is equal to the number of customers of these nodes and is represented by the vector:

$$\mathbf{g}^T = (112; 115; 122; 122; 119; 108; 88; 109)$$

The numbers are given in thousands and are rounded. This means that, for example, in the first node (the city of Banská Bystrica) there are currently 160 thousand potential customers, whom companies can get and sell their products to. In the second node (Bratislava) there are 33 thousand more of them, that means 115 thousand customers. This continues until the last, eighth node, which corresponds to the city of Žilina and where there are currently 165 thousand potential customers.

Let us consider situation, where transportation costs are at level $t = 0.2$. The matrix $\mathbf{D} = d(i,j)$, $i,j \in V$, representing the shortest distances between individual regional cities has following form:

$$\mathbf{D} = \begin{bmatrix} 0 & 207 & 213 & 119 & 248 & 142 & 165 & 89 \\ 207 & 0 & 420 & 88 & 419 & 125 & 47 & 198 \\ 213 & 420 & 0 & 332 & 35 & 329 & 378 & 256 \\ 119 & 88 & 332 & 0 & 361 & 85 & 46 & 140 \\ 248 & 419 & 35 & 361 & 0 & 294 & 372 & 221 \\ 142 & 125 & 329 & 85 & 294 & 0 & 78 & 73 \\ 165 & 47 & 378 & 46 & 372 & 78 & 0 & 151 \\ 89 & 198 & 256 & 140 & 221 & 73 & 151 & 0 \end{bmatrix}$$

Let us have situation in which the pre-known unit price of the other player's product will be $p_2 = 100$. The product price of the player 1 must be within interval $\langle p_1^{lo}; p_1^{up} \rangle$ where $p_1^{lo} = 50$ and $p_1^{up} = 150$. This means that he knows that his price cannot differ from the other player's price by more than 50%. The situation is described by model (10)-(16), (18), (19) and (26).

Calculated price of player 1 would then be at level $p_1^* = 90.799$, at which he achieves revenues of 59,525.292, while serving almost 656 customers. In this case, his strategy is given by the vector $\mathbf{x}^T = (0; 0; 0.006; 0.110; 0; 0.426; 0; 0.458)$.

Now we can extend the assumptions to costs associated with location $h = 5{,}000$. Using model (11)-(16), (18), (19), (22), (23) and (26) we obtain following results.

The payment matrix **A** will have the form:

$$\mathbf{A} = \begin{bmatrix} 895 & 692 & 654 & 570 & 654 & 699 & 692 & 786 \\ 433 & 895 & 654 & 539 & 654 & 437 & 895 & 433 \\ 241 & 462 & 895 & 241 & 895 & 353 & 353 & 241 \\ 542 & 780 & 654 & 895 & 654 & 895 & 895 & 667 \\ 241 & 462 & 895 & 350 & 895 & 241 & 462 & 241 \\ 661 & 780 & 654 & 895 & 654 & 895 & 895 & 895 \\ 542 & 895 & 654 & 895 & 654 & 895 & 895 & 545 \\ 783 & 692 & 654 & 570 & 654 & 895 & 570 & 895 \end{bmatrix}$$

In case of additional costs for the players, the player 1 will set his price to $p_1^* = 82.999$, at which he will achieve revenue of 42,224.405 serving 689.459 units of the demand. Equilibrium strategies of the players are represented by vectors $\mathbf{x}^T = (0.386; 0; 0; 0; 0.147; 0.467; 0; 0)$ and $\mathbf{y}^T = (0.196; 0; 0.662; 0.141; 0; 0; 0; 0)$. It is obvious, that the solution does not represent saddle point of the matrix.

Let us now introduce an assumption of limited supply for player 1. He knows that he cannot serve more than 600 units of demands, but he does not know the capacity of the opponent (he considers it to be large enough). The introduction of this assumption will change results as follows:

Using equations (10)-(16), (18), (19), (26) and (27) we obtain new calculated price of player 1 at level $p_1^* = 99.999$. At such price the player serves 521.721 units of demand and reaches revenue of 52,172.095. His equilibrium strategy is given by vector $\mathbf{x}^T = (0; 0; 0; 0.020; 0; 0.436; 0; 0.544)$, so he should invest 1.20% in node 4, 43.6% in node 6 and 54.4% in node 8.

If we added operating costs of $h = 5{,}000$ into the problem, using model (11)-(16), (18), (19), (22), (23), (26) and (27) we obtain the payment matrix **A**:

$$\mathbf{A} = \begin{bmatrix} 600 & 600 & 600 & 570 & 600 & 584 & 600 & 600 \\ 433 & 600 & 600 & 539 & 600 & 437 & 600 & 433 \\ 241 & 462 & 600 & 241 & 600 & 353 & 353 & 241 \\ 542 & 600 & 600 & 600 & 600 & 600 & 600 & 545 \\ 241 & 462 & 600 & 241 & 600 & 241 & 350 & 241 \\ \mathbf{600} & \mathbf{600} & \mathbf{600} & \mathbf{600} & \mathbf{600} & \mathbf{600} & \mathbf{600} & \mathbf{600} \\ 542 & 600 & 600 & 600 & 600 & 559 & 600 & 433 \\ 600 & 600 & 600 & 570 & 600 & 600 & 570 & 600 \end{bmatrix}$$

The weak solution is represented by 8 saddle points (all marked in matrix **A**), while equilibrium strategy of player 1 is $x_6 = 1$. He sets his price to the level of $p_1^* = 85.4$ in this case, serves 600 units of demand, which is equal to the limit of his supply, and achieves revenue of $46{,}240$.

If the operating costs increased to $h = 7{,}000$, the solution would remain the same as in the previous case. The additional costs would only change the final revenue, which would fall by 2.

6. Conclusion

This article was focused on the analysis of a specific spatial game, which deals with placement on a graph. The demand, located in the nodes, is divided among duopolists, who decide on the location of their service point. The duopolists compete to attract their potential customers, who choose one of them according to their total costs, consisting of product price and shipping costs. To analyze this situation can be used constant-sum games.

In this article we address the specific problem where the price of product of one duopolist is known in advance, which means that the price is variable in the model and the other duopolist tries to set his price to maximize his revenues based on the best response. Unlike most of the literature dealing with the issue of spatial models of oligopoly, focused on sequential models, where locations are selected in first step and the prices in the second, we present models, where both problems are solved at the same time.

The article was structured as follows: the main parts of the article are focused on the original mathematical models falling under mixed integer nonlinear programming (MINLP) allowing to determine the price of one of the duopolists based on best response. The presented model considers the unit demand of each node. In the next part we extended the basic assumption by the introducing of different demands for nodes and subsequently with the assumption of limited supply of the duopolist who sets his price. We present both situations on illustrative examples. We verify the results using Fixed price model, in which both prices are known in advance. In the last part we presented the solution of problem where nodes represent regional cities of Slovak Republic, while the demands of these cities are related to the number of their inhabitants. We consider the assumption of an unlimited offer of both duopolists and also the case of a limited offer of one of them.

There are many ways for further extension of the presented approach. One possibility is to introduce an assumption of lost demand with a specified upper limit on the costs that consumers are willing to bear in connection with the procurement of goods. Such situation can be modelled using games with nonconstant-sum games. We also see another direction of development in the area of regulation, when the regulator intervenes in the mentioned situation, with the aim of preference of some localities, or by influencing the final prices of producers. An equally important is the area of computability. This model falls under the MINLP. Although the presented calculations were performed in real time (GAMS software, Couenne solver 25.1.3 on a PC with Intel® Core® i7-8650U CPU @ 1.9 GHz 2.10 GHz with 16 GB RAM), the computability of larger problems could be complicated. Therefore, another possible extension of the presented approach could be a suitable transformation of the model or the design of an effective algorithm for solving such problems.

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