

A Duration Dependent Rating Migration Model: Real Data Application and Cost of Capital Estimation*

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Abstract

This paper presents a duration dependent model for analyzing the evolution of credit ratings. It considers the backward recurrence process to tackle the time of permanence problem in the rating classes. In this way it is possible to manage the duration effects, which represent one of the most important features in rating dynamics. Furthermore, the paper shows how it is possible to compute the cost of capital that an organization is required to pay for the capital used in financing its activities. A real data application using Standard & Poor's historical database is provided.

1. Introduction

The main purpose of this paper is to propose a new stochastic model that is able to analyze credit rating migrations and to evaluate the cost of capital necessary for financing a company's activity.

The problem of credit risk basically consists of computing the probability that a firm will go into the default state. The literature on this topic is copious and the interested reader can refer to books by Chacko et al. (2006), Lando (2004) and Bluhm et al. (2002) or to papers by Carty and Fons (1994), Nickell et al. (2000), Bangia et al. (2002), Řezáč and Řezáč (2011), Jagric et al. (2011) and Gapko and Šmíd (2012).

International organizations, such as Fitch, Moody's and Standard & Poor's (S&P), evaluate credit risk by giving different ratings to firms that have agreed to be inspected. Such firms receive a note (rating) that represents an assessment of how reliable they are in terms of repaying their own debt. The rating is a measure of the creditworthiness of bonds issued. Clearly, the lower the rating, the higher the interest rate that the firm should pay.

The rating changes with time and a way of following its evolution is by means of Markov processes (see, for example, Jarrow et al., 1997; Israel et al., 2001; Hu et al., 2002). In this environment, Markov models are called *migration models*.

The problem of the poor fitting of Markov processes in the credit risk environment has been outlined in some papers (see, for example, Altman, 1998; Carty and Fons, 1994; Lando and Skodeberg, 2002). More efficient rating migration models based on semi-Markov processes have been proposed recently (e.g. D'Amico et al.,

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2005; Vasileiou and Vassiliou, 2006; Vasileiou and Vassiliou, 2013; D'Amico et al., 2009a; D'Amico et al., 2010a). The semi-Markov models represent a generalization of the Markovian models that allow the possibility to use any kind of distribution functions for modeling the sojourn times in the states of the system.

In the article by D'Amico et al. (2005), a model based on homogeneous semi-Markov processes was presented. In that paper the duration problem was considered for the first time, although in an incomplete way. The duration effect affirms that the time during which a company is in a rating class influences the transition probabilities and, consequently, the company's reliability functions.

A generalization of the transition probabilities for a non-homogeneous semi-Markov process (NHSMP) is obtained by introducing the backward environment (see D'Amico et al., 2009b). In this case, the transition probabilities are also dependent on the time of entrance into a given state. For this reason, this paper proposes a model based on an NHSMP that efficiently solves the credit rating problem.

The rating evolution is investigated by using real data from the rating agency S&P and, for the first time, a duration dependent rating migration model is implemented to describe the duration effect implicit in real data. The classical rating models consider the No Rating class (NR) for unrated companies. Since the reasons for an NR assessment may be positive or negative, as better explained later in this paper, the proposed model considers the splitting of the NR state into two artificial rating classes, called NR1 and NR2, associated with positive and negative causes, respectively.

Moreover, the proposed model permits quantification of the cost of capital that an organization is required to pay for the capital used in financing its activities. This paper proves that the cost of capital, depending on the transition probabilities and reliability functions with initial and final backward times, is also influenced by the duration inside the rating classes.

Therefore our suggestion is that the duration effects should be taken into account in the classical cost of capital analyses and investment decisions (see, for example, Magni, 2011; Block, 2011; Gurgur, 2011).

2. Duration Dependent Semi-Markov Models

In this section semi-Markov models, which take into account the initial and final backward times, are briefly described; for more details, see D'Amico et al. (2009b). In a semi-Markov process environment, two random variables evolve jointly. J_n , $n \in \mathbb{N}$ with state space $I = \{1, 2, \dots, m\}$ represents the state at the n -th transition. T_n , $n \in \mathbb{N}$ with state space equal to \mathbb{N} represents the time of the n -th transition.

The process (J_n, T_n) is supposed to be a non-homogeneous Markov renewal process with kernel $\mathbf{Q} = [Q_{ij}(s, t)]$ defined in the following way:

$$Q_{ij}(s, t) = \mathbb{P}[J_{n+1} = j, T_{n+1} \leq t \mid J_n = i, T_n = s] \\ = \mathbb{P}[T_{n+1} \leq t \mid J_n = i, J_{n+1} = j, T_n = s] \cdot \mathbb{P}[J_{n+1} = j \mid J_n = i, T_n = s] =: G_{ij}(s, t) \cdot p_{ij}(s)$$

The main difference between a non-homogeneous Markov process and an NHSMP is in the probability distribution function $G_{ij}(s, t) = \mathbb{P}[T_{n+1} \leq t \mid J_n = i, J_{n+1} = j, T_n = s]$.

In a Markovian environment, this function has to be a geometric distribution function; however, in the semi-Markovian case it can be of any type.

The $p_{ij}(s)$ are the transition probabilities of the non-homogeneous embedded Markov chain J_n . Now let $N(t) = \max\{n \in \mathbb{N} \mid T_n \leq t\}$ then the NHSMP $Z(t) = J_{N(t)}$, $t \in \mathbb{N}$ can be defined as the state occupied by the process at each time. The transition probabilities are defined in the following way: $\phi_{ij}(s, t) = P[Z(t) = j \mid Z(s) = i, T_{N(s)} = s]$.

They are obtained by solving the following evolution equations:

$$\phi_{ij}(s, t) = \delta_{ij} \left(1 - \sum_{j \neq i} Q_{ij}(s, t) \right) + \sum_{\beta=1}^m \sum_{\vartheta=s+1}^t (Q_{i\beta}(s, \vartheta) - Q_{i\beta}(s, \vartheta-1)) \phi_{\beta j}(\vartheta, t) \quad (1)$$

where $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$.

The first addendum of formula (1) gives the probability that the system does not have transitions up to time t given that it entered state i at time s . The second addendum represents the probability that the system enters state β just at time ϑ , given that it entered state i at time s and then, after the transition, will follow one of the possible trajectories connecting state β at ϑ , to state j at t .

In order to appropriately consider the duration problem, it is necessary to introduce the backward time recurrence process: $B(t) = t - T$.

The process $B(t)$ denotes the time since the last transition (Limnios and Oprışan, 2001; Janssen and Manca, 2006; Janssen and Manca, 2007).

As derived in D'Amico et al. (2009b), denoted by:

$${}^b\phi_{ij}(v, s; t) = P[Z(t) = j \mid Z(s) = i, B(s) = s - v] \quad (2)$$

the semi-Markov transition probabilities with initial backward times that express the probability of staying in state j at time t , given that at time s the system was in state i , with entrance into this state at time v . The difference $s-v$ represents the initial backward time (the duration in state i).

The duration effect also occurs when the system enters state j : the system can enter state j at time v' and never exit from it up to time t . For this reason formula (2) can be further generalized:

$${}^b\phi_{ij}^b(v, s; v', t) = P[Z(t) = j, B(t) = t - v' \mid Z(s) = i, B(s) = s - v] \quad (3)$$

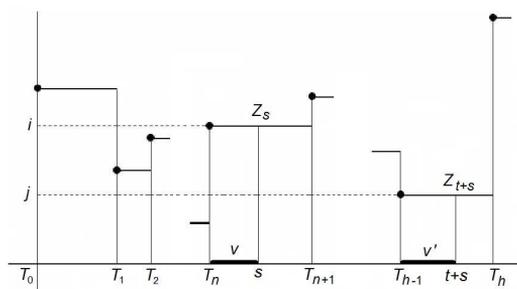
A trajectory of an NHSMP with initial and final backward times is reported in Figure 1. Here we have that $N(s) = n, N(t) = h-1$, the initial backward $B(s) = s - T_n = s - v$ and the final backward value $B(t) = t - T_{h-1} = t - v'$.

The transition probabilities (2) and (3) satisfy recursive renewal type equations that generalize equation (1) (see D'Amico et al., 2009b).

3. Duration Dependent Rating Migration Model

The credit risk migration problem can be studied using an NHSMP. The rating process, carried out by the rating agency, gives a measure of the creditworthiness of

Figure 1 A Trajectory of a NHSMP with Initial and Final Backward Recurrence Times



a particular company, security or obligation. The rating agency S&P basically considers eight different classes of *long-term rating*. They are listed to form the following coarse set of states: $I = \{AAA, AA, A, BBB, BB, B, CCC, NR, D\}$.

The rating AAA indicates the highest level of creditworthiness, assigned to a financial instrument considered extremely reliable with regard to financial obligations, and decreases towards the rating CCC, which expresses a bankruptcy petition that has been filed or similar action taken, but payments of financial commitments have continued.

A table showing the financial meaning of the S&P rating categories is reported in Bluhm et al., 2002.

The ratings from AA to CCC may be modified by the addition of a plus or minus sign to show relative standing within the major rating categories and then a finer set of ratings could be considered. Moreover, S&P also publishes rating outlooks (positive, neutral, negative) that assess the potential for change by incorporating trends or risks with certain implications for credit quality (see Trueck and Rachev, 2009). It is possible to consider the outlook effects within the proposed model simply by enlarging the state space and by considering each state as a couple (rating-outlook). Anyway, these strategies imply a dramatic increase in the number of parameters to be estimated. The estimation of a semi-Markov kernel requires large numbers of observations of rating transitions even for the coarse rating categories. For this reason, and based on previous studies on the Markov chain migration model (e.g. Hu et al., 2002; Crouhy et al., 2001) only the coarse scale is considered in the application.

An issue designated NR is not rated. In other words, at some point during its rating history it had its rating withdrawn and is consequently removed from consideration, but is monitored with the aim of capturing a potential default. Ratings are withdrawn when rated programs are terminated and the relevant debt is extinguished or when the entity leaves the public fixed-income markets. Moreover mergers, acquisitions, lack of cooperation, insufficient information, matter of policy and redemptions are all reasons warranting an NR rating.

State D expresses the occurrence of payment default on a financial obligation. Since reasons for an NR assessment may be both positive and negative, a splitting of the NR state into two new states denoted by NR1 and NR2 is recommended (D'Amico et al., 2010a). An instrument is rated NR1 when it enters state NR coming from an investment grade. The term *investment grade* refers to instruments ranked up

to BBB key. This term is broadly used to describe instruments with relatively high levels of creditworthiness and credit quality. In contrast, an instrument is rated NR2 when it enters state NR coming from a speculative grade. Non-investment or speculative grade refers to debt securities, where the issuer has the ability to repay but faces significant uncertainties that could affect credit risk. The related ratings are BB or lower. Obviously, this higher default risk is compensated by a higher possible return.

As usual in reliability theory, the state space is partitioned into two subsets denoted by U and D . Subset U contains all of the good ratings (up states) and subset D contains all of the bad ratings (down states):

$$U = \{AAA, AA, A, BBB, BB, B, CCC, NR1\}; \quad D = \{\overline{NR2}, D\} \quad (4)$$

This approach allows us to use the classical reliability indicators such as the availability, reliability and maintainability functions. These indicators have been generalized by also taking into account the initial and final backward and forward recurrence times in D'Amico et al. (2010b). Below, these indicators are evaluated by considering the initial and final backward times on the real data set.

The availability with initial and final backward times is defined as follows:

$${}^b A_i^b(v, s; v', t) := P[Z(t) \in U, B(t) = t - v' | Z(s) = i, B(s) = s - v] = \sum_{j \in U} {}^b \phi_{ij}^b(v, s; v', t) \quad (5)$$

This gives the probability that the instrument has an up rating at time t whatever happens at $(s, t]$ considering the initial and final backward times.

The reliability with initial and final backward times is defined $\forall i \in U$ as follows:

$$\begin{aligned} {}^b R_i^b(v, s; v', t) &:= P[Z(h) \in U : \forall s < h \leq t, B(t) = t - v' | Z(s) = i, B(s) = s - v] \\ &= \sum_{j \in U} {}^b \tilde{\phi}_{ij}^b(v, s; v', t) \end{aligned} \quad (6)$$

where ${}^b \tilde{\phi}_{ij}^b(v, s; v', t)$ are the transition probabilities with initial and final backward times computed by using the following kernel transformation:

$$\tilde{p}_{ij} = \delta_{ij} \quad \forall i \in D \quad (7)$$

The transformation (7) defines a new kernel $\tilde{\mathbf{Q}}$ for which all the states of the subset D are changed in absorbing states.

Formula (6) gives the probability that the instrument always has an up rating from time s to time t considering the initial and final backward times. Notice that the reliability with initial backward time in the following equation:

$${}^b R_i(v, s; t) := \sum_{v'=v}^t {}^b R_i^b(v, s; v', t) \quad (8)$$

has to be introduced into the analysis in relation to the cost of capital estimation.

The maintainability with initial and final backward times is defined $\forall i \in D$ as follows:

$$\begin{aligned}
 {}^b M_t^b(v, s; v', t) &:= 1 - P[Z(h) \in D, \forall s < h \leq t, B(t) = t - v' | Z(s) = i \in D, B(s) = s - v] \\
 &= \sum_{j \in U} {}^b \bar{\phi}_{ij}^b(v, s; v', t)
 \end{aligned} \tag{9}$$

where ${}^b \bar{\phi}_{ij}^b(v, s; v', t)$ are the transition probabilities with initial and final backward times computed by using the following kernel transformation:

$$\bar{P}_{ij} = \delta_{ij} \quad \forall i \in U \tag{10}$$

The transformation (10) defines a new kernel $\bar{\mathbf{Q}}$ for which all the states of the subset U are changed in the absorbing states.

Formula (9) gives the probability that the instrument will have an up rating at least once from time s to time t when considering the initial and final backward times.

The choice of a non-homogeneous model permits taking into consideration periods of financial and economic crises characterized by an increase in default rates. Indeed, in a non-homogeneous model, transition probabilities and credit risk indicators are time varying and can be computed for each time couple s, t .

4. Cost of Capital Implications

In this section the impact of ratings on the cost of capital and the implication of a duration dependent rating model are considered. To this end, let us consider a firm that at current time s has the rating $Z(s) = i$ of duration $B(s) = v$. Seeking a source of financing, it decides to issue a financial obligation with inception time t and maturity $t + x$ promising to pay EUR 1.

The interest rate the firm will pay for this contract depends on the firm's reliability, which gives important financial information. For example, if the one-period reliability is 0.98 (the probability of being reimbursed after one period) and the one-period free risk interest rate is 0.03, then the obligation should pay the following one-period rate of interest: $\frac{1.03}{0.98} - 1 \approx 5.1\%$.

The interest rate the firm will pay for the considered contract depends on the credit rating $Z(t)$ and on the duration $B(t)$ because the reliability function in our model is $(Z(t), B(t))$ -dependent. At each time s , these quantities are random variables.

Therefore, if the one-period risk free interest rate is r , then the capitalized value of EUR 1 during the next period is

$$\frac{(1+r)}{{}^b R_{Z(t)}(t - B(t), t; t + x)^{\frac{1}{x}}} \tag{11}$$

The duration dependent rating model allows us to compute the expected cost of financing by evaluating the expectation of the random variable (11):

$$E_s \left[\frac{(1+r)}{\left({}^b R_{Z(t)}(t-B(t), t; t+x) \right)^{\frac{1}{x}}} \right] = \sum_{j \in U} \sum_{d=v}^t \frac{{}^b \phi_{ij}^b(v, s; d, t)}{{}^b R_i(v, s; t)} \left[\frac{(1+r)}{\left({}^b R_j(d, t; t+x) \right)^{\frac{1}{x}}} \right] \quad (12)$$

where $E_s[X]$ is the expectation of random variable X evaluated by using the information at time s . The variable (11) takes the value $\frac{(1+r)}{{}^b R_j(d, t; t+x)^{\frac{1}{x}}}$ with proba-

bility: $P[Z(t) = j, B(t) = t - d | Z(s) = i, B(s) = s - v, Z(t) \in U] = \frac{{}^b \phi_{ij}^b(v, s; d, t)}{{}^b R_i(v, s; t)}$.

The riskiness of the contract can be evaluated by computing the variance of (11):

$$V_s \left[\frac{(1+r)}{\left({}^b R_{Z(t)}(t-B(t), t; t+x) \right)^{\frac{1}{x}}} \right] = \sum_{j \in U} \sum_{d=v}^t \frac{{}^b \phi_{ij}^b(v, s; d, t)}{{}^b R_i(v, s; t)} \left[\frac{(1+r)}{\left({}^b R_j(d, t; t+x) \right)^{\frac{1}{x}}} \right]^2 - \left[\sum_{j \in U} \sum_{d=v}^t \frac{{}^b \phi_{ij}^b(v, s; d, t)}{{}^b R_i(v, s; t)} \left[\frac{(1+r)}{\left({}^b R_j(d, t; t+x) \right)^{\frac{1}{x}}} \right] \right]^2$$

By means of similar arguments the skewness and kurtosis can also be computed.

5. The Credit Risk Model and the Related Algorithm

5.1 Non-Parametric Estimation of Indicators

In order to achieve the claimed results, it is necessary to estimate the semi-Markov kernel $\mathbf{Q} = [Q_{ij}(s, t)]$ and the proposed credit rating indicators from the considered data set. Firstly, the following symbols are introduced:

- K is the number of independent trajectories in the data set, which denotes the credit rating histories of the rated financial instruments.
- J_n^r is the rating at the n -th transition of the r -th financial instruments.
- T_n^r is the time in which the r -th financial instrument makes the n -th change of rating.
- $N^r = N^r(T) = \sup\{n \in N : T_n^r \leq T\}$ is the total number of transitions held by the r -th financial instrument.
- $N_i^r = N_i^r(T) = \sum_{k=1}^{N^r} \mathbf{1}_{\{J_{k-1}^r = i\}}$ is the number of visits of the r -th financial instrument to rating class i .

- $N_i = N_i(T) = \sum_{r=1}^K N_i^r$ is the total number of visits of all financial instrument to rating class i .

Then consider the following empirical kernel estimator on the analogous estimator defined by Oubhi and Linnios (1999) in the homogeneous case:

$$\hat{Q}_{ij}(s, t, K) = \frac{1}{N_i} \sum_{r=1}^K \sum_{l=1}^{N^r} \mathbf{1}_{\{J_{l-1}^r=i, J_l^r=j, T_{l-1}^r=s, T_l^r \leq t\}} \quad (13)$$

After evaluating estimator (13) it is possible to estimate transition probabilities, credit rating indicators and the cost of capital measures by using a plug-in procedure, which involves replacing the theoretical kernel $Q_{ij}(s, t)$ with its empirical estimators $\hat{Q}_{ij}(s, t, K)$ in the indicators' computation.

Due to lack of space, the kernel estimates are not provided here, but they are available upon request.

5.2 The Algorithm

The relationship between a discrete time initial and final backward semi-Markov process is fully described above and the program is presented by means of a pseudolanguage in the *Appendix* on the web-site of this journal, so that the reader can follow the algorithm used. The implementation of the algorithm requires the introduction of some auxiliary variables necessary to compute ${}^b\phi_{ij}^b(v, s; v', t)$ (see D'Amico et al., 2009b).

After computing the probabilities ${}^b\phi_{ij}^b(v, s; v', t)$, it is possible to evaluate the availability with initial and final backward times simply by summing the ${}^b\phi_{ij}^b(v, s; v', t)$ on the states $j \in U$; see formula (5).

Next, in order to evaluate the reliability ${}^bR_i^b(v, s; v', t)$, it is sufficient to replace matrix $\mathbf{P}(s)$ with $\tilde{\mathbf{P}}(s)$, whose elements are given in formula (7), and then run the algorithm again.

Finally, in order to evaluate the maintainability ${}^bM_i^b(v, s; v', t)$, it is sufficient to replace matrix $\mathbf{P}(s)$ with $\tilde{\tilde{\mathbf{P}}}(s)$, whose elements are given in formula (10), and then run the algorithm again.

5.3 Application to Standard & Poor's Historical File

Data are taken from the S&P rated universe. The database refers to files concerning the entity ratings history, instrument ratings history and issue/maturity ratings history. The history starts in 1922 and ends on 31 December 2008. In view of maintaining statistical significance, only the rated financial instruments (instrument history and issue/maturity history) from 1981 are selected.

The terms *instrument* and *issue/maturity* include any kind of individual debt issued by the entities at a particular time and refer to specific maturities and/or programs.

Only the long-term ratings assessed by S&P to instruments are considered. They express an opinion about the credit quality of an individual debt instrument and take into consideration the creditworthiness of guarantors, insurers and other forms of credit enhancement based on the obligation and also the currency in which the obligation is denominated.

The considered time scale is one year. This choice agrees with the majority of credit rating migration studies (see, for example, Carty and Fons, 1994; Bangia et al., 2002; Jafry and Schuermann, 2004). Moreover, as highlighted by Bangia et al. (2002), “the other factor determining the transition horizon is the application purpose”. For the calculation of the cost of capital, a one year transition horizon is standard. Without additional computational difficulties, a more granular (monthly or quarterly) time scale could be adopted for measuring the eventual rating transitions of a financial instrument within one year. Anyway, a finer time scale requires a larger dataset to non-parametrically estimate the waiting time distribution functions $G_{ij}(s;t)$, because the probability distribution functions are spread out on finer supports requiring the estimation of more probability masses. At the same time, since the rating migrations of a single financial instrument are not very frequent, the transitions within one year are not sufficiently informative to allow significant estimation of the increased number of probability masses. Therefore, the choice of the one year time scale is connected to the choice of a non-parametric model due to the low frequency of the rating changes. Anyway, it should be stated that the results might be different in the case of more granular data and that shorter horizons such as semi-annual or quarterly tend to have results similar to those obtained by the continuous time model as in Lando and Skodeberg (2002) and Jafry and Schuermann (2004). Then, this paper focus on the one-year horizon, as that is typical for many credit applications, cost of capital included.

Table 1 in the Appendix shows the one-year instruments frequencies subdivided by rating class. *Figure 2* reports the number of transitions that occurred among the ratings from 1981 to 2008.

A huge amount of results was obtained and only some of them are included here; however, they are all available upon request. The transition probabilities of the embedded Markov chain at $s = 0$ are reported in *Table 2 in the Appendix*.

A look to the table reveals some interesting features. First of all, rating D is not an absorbing state. Indeed, there is probability equal to 0.909 of entering state NR2 with the next transition; the remaining probability mass is spread over the speculative grades. This behavior is repeated over time but with different transition probabilities.

Next, entry into state NR1 is more likely from higher investment grades. The opposite happens for entry into state NR2: the worse the rating, the more probable the entry into state NR2. Finally, the analysis of all the transition matrices of the embedded Markov chain suggests that the adoption of a non-homogeneous model is more appropriate because of the different transition probability values depending on the years.

Figure 2 Number of Rating Transitions per Year

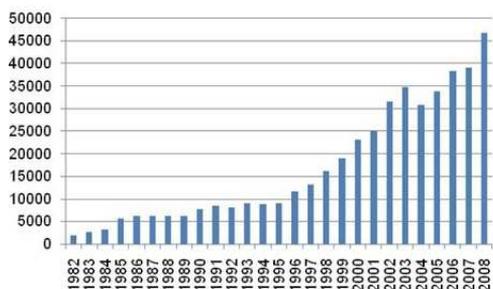
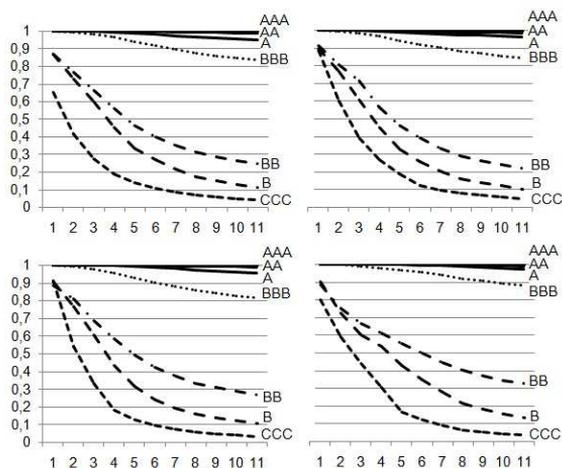


Figure 3 Reliability Functions with Initial Backward ${}^bR_i(v, s; s+h)$



Notes: On the x-axis there are the h -values; on the y-axis there are the probabilities. Top-left panel shows the case $v = 8, s = 8$, top-right panel shows the case $v = 7, s = 8$, bottom-left panel shows the case $v = 6, s = 8$, bottom-right panel shows the case $v = 4, s = 5$.

The semi-Markov transition probabilities with initial backward times for different values of v, s and t are reported in the *Appendix* (see *Table 3 in the Appendix*).

Notice that the difference $(t-s)$ is equal to four years in all cases considered. The comparison between the first three cases shows differences that can be imputable to non-homogeneity, because they have the same initial backward value of 1.

The comparison between the last two cases shows differences that can be imputable to the different initial backward values, because the values s and t are the same.

Now, as an example, if the results related to times $(2,3;7)$ and $(10,11;15)$ are compared, then the differences are imputable to both the non-homogeneity effect and to the initial backward values.

In *Figure 3* the reliability functions with initial backward times for different values of v, s and t obtained by ${}^bR_i(v, s; t) = \sum_{v'=0}^{t-v} {}^bR_i^b(v, s; v', t)$ are shown.

From Table 4 (5) in the Appendix, it is possible to take over the dependence of the reliability (maintainability) functions on the initial and final backward times. In fact, by comparing two columns, it is possible to see different reliability (maintainability) values due to different initial backward times. By comparing two rows, it is possible to see the effects due to different final backward values.

The model also permits the analysis concerning the cost of capital depending on the riskiness of the firm. To this end, consider the example of a firm that, at current time $s = 6$, has the rating $Z(s) = i$ of duration $B(s) = s - v = 1$. Seeking a source of financing, it decides to issue a financial obligation with inception time $t = 7$ and maturity $t + x = 8$ promising to reimburse EUR 1 at the maturity date. The computation of the expectation (12) produces the results given in Table 6 in the Appendix.

It is possible to observe the monotonicity in the capitalized values with respect to the different rating classes. Concerning the *investment grade classes*, the capitalized values range from 1.030 to 1.033. The differences inside the investment ratings are expected to increase if a maturity duration longer than one year is considered. In contrast, the capitalized values are considerably burdensome for the *speculative grade classes*, ranging from 1.194 to 1.224.

6. Conclusions

In this paper a discrete time non-homogeneous semi-Markov process with initial and final backward recurrence times was considered in order to execute a real application to credit rating data. The model was applied to data from the historical database of Standard & Poor's from 1981 to 2008 and was considered over an annual time scale. The raw data were reorganized with the aim of better describing the financial meaning of the No Rating class (NR), which was then split into two new states: one *good* and the other *bad*. The introduction of such artificial ratings guarantees the monotonicity of the reliability function with respect to the rating classes.

It is worth noting that the introduction of recurrence times is necessary in order to capture and describe the duration effects in rating models. The duration effect is crucial because the time of permanence in a credit rating changes the transition probabilities and the reliability functions. The application results were very interesting because they confirmed the duration effects, namely that changing the initial and final backward times may alter the results considerably.

A quantification of the cost of capital that an organization is required to pay for the capital used in financing its activities was performed. The paper proved that the cost of capital, depending on the transition probabilities and reliability functions with initial and final backward times, is also influenced by the duration within a rating class.

Further developments include the re-estimation of the model using additional information, such as a more detailed structure of rating classes, rating outlooks and more granular (monthly or quarterly) datasets. These developments are conditional on the acquisition of a larger dataset, which will be certainly available in the near future given the growth of the rating market.

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