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Backtesting of portfolio optimization with and without risk-free asset

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Abstract

A classical question in modern portfolio theory asks how the best portfolio composition can be chosen. Answering this question is definitely not easy and the general approach is to maximize the chosen risk-reward ratio. In our paper, however, we utilize the mean-variance framework introduced by Markowitz and maximize the (quadratic) utility function, which depends on the expected return (future mean return) and its variance. Simplification in terms of the applied utility function instead of the performance ratio allows portfolio backtesting over a relatively long period with a short computation time. The goal of the paper is to analyse how the risk-free asset investment possibility influences the ex post observed wealth path in the case of the selected period and data set. We find that the possibility of risk-free investments actually deteriorates the wealth path. Our explanation is that the simple portfolio optimization strategy proposed in the paper is unable to forecast market declines in time and reacts with a delay.

Keywords

Backtesting, portfolio optimization, wealth maximization.

JEL Classification: G11, G17

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Backtesting of portfolio optimization with and without risk-free asset

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1. Introduction

A classical question in modern portfolio theory asks how the best portfolio composition can be chosen. Answering this question is definitely not easy and the general approach is to maximize the chosen risk-reward ratio. Based on the applied risk and reward measures, plenty of performance ratios are available. Examples are the Sharpe ratio (Sharpe, 1966, 1994), Gini ratio (Shalit and Yitzhaki, 1984), mean absolute deviation ratio (Konno and Yamazaki, 1991), mini-max ratio (Young, 1998), Rachev ratio (Biglova et al., 2004) and others; for a summary, see for example Farinelli et al. (2008). In our paper, however, we utilize the mean-variance framework introduced by Markowitz (1952) and maximize the quadratic utility function, which depends on the expected return (future mean return) and its variance. Simplification in terms of the applied utility function instead of the performance ratio allows portfolio backtesting over a relatively long period with a short computation time due to the ease of finding the global optimum. For more complex applications assuming the maximization of different performance ratios, see for example Cassader et al. (2014), Petronio et al. (2014) or Giacometti et al. (2015).

The goal of the paper is to analyse how the risk-free asset investment possibility influences the ex post observed wealth path in the case of the selected period and data set. The analysis is performed by means of backtesting two portfolio optimization problems: with and without the possibility of risk-free investments. The backtesting is performed on a data set that consists of the stocks incorporated into the Dow Jones Industrial Average index over the period from 30 December 1994 to 31 December 2014.

The paper is structured as follows. In the next section, the portfolio optimization models in Markowitz's mean-variance framework are described. Then, in the third section, we briefly describe the backtesting procedure that we apply. In the fourth section, the empirical results are presented. The last two sections provide a discussion of the results and the conclusion.

2. Portfolio optimization problem

The cornerstone of modern portfolio theory was established by the pioneering work of Harry Markowitz in 1952 reported in his well-known paper; see Markowitz (1952). We assume a portfolio composed of N assets, for which the joint probability distribution of returns is known, that is, we know both the expected returns of particular assets $E(r_i)=\left[E(r_1),...,E(r_N)\right]^T$ and the covariance matrix of the returns $Q=\left[\sigma_{i,j}, i=1,..,N\ j=1,..,N\right]$. Then, assuming the portfolio composition $x=\{x_1,...,x_N\}^T$ and based on the normality assumption, we can compute the portfolio expected return $E(r_p)$ and the portfolio variance σ_p^2 (standard deviation σ_p, respectively) as follows:

$$E(r_p)=\sum_{i=1}^{N} x_i \cdot E(r_i) = x^T \cdot E(r_i), \tag{1}$$

$$\sigma_p^2=\sum_{i=1}^{N}\sum_{j=1}^{N} x_i \cdot \sigma_{i,j} \cdot x_j = x^T \cdot Q \cdot x, \tag{2}$$

$$\sigma_p=\sqrt{\sigma_p^2}. \tag{3}$$

The rational investor wants to maximize the portfolio expected return and (assuming that he is risk averse) minimize the variance. However, the relationship between these two characteristics is generally positive – by lowering the variance, the expected return also decreases; see for example Lundblad (2007).

Without the knowledge of the investor's risk aversion, we can find the set of (Pareto) efficient portfolios. A portfolio is identified as efficient if the following conditions are fulfilled at the same time:

- All the other portfolios that have the same (or a higher) expected return also have higher variance,
- all the other portfolios that have the same (or a lower) variance have a lower expected return.

The above can be reversed so that the particular portfolio is efficient if and only if there is no other portfolio with lower variance delivering a higher or equal expected return and no other portfolio with a higher expected return and lower (or equal) variance. For further details of modern portfolio theory, see for example Elton et al. (2009). Generally, other limitations can also be imposed; for instance, see the cardinality constrained portfolio optimization problem in Salahi et al. (2014).

We further assume that we know the level of the investor's risk aversion. We assume an investor whose utility function can be formulated as follows:

$$U = E\left(r_p\right) - k \cdot \sigma_p^2, \tag{4}$$

meaning that this investor is maximizing the expected return from the portfolio while minimizing the risk. The risk-return trade-off that the investor is capable of undertaking while maintaining the same utility is depicted by parameter *k*. Every increase in the variance has to be compensated for by a *k*-times increase in the expected return to keep the same level of utility. Investors with different risk attitudes can be distinguished according to the value of parameter *k*: i) risk-averse investors can be characterized by positive values of the parameter, ii) risk lovers are connected to negative values and iii) *k = 0* for risk-neutral investors.

The rational investor seeks the highest utility possible. Thus, we can formulate the following maximization problem:

$$w = \begin{cases} \arg\max\limits_{x} \sum\limits_{i=1}^{N} x_i \cdot E\left(r_i\right) - k \cdot \sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} x_i \cdot \sigma_{i,j} \cdot x_j \\ \sum\limits_{i=1}^{N} x_i = 1 \\ x_i \geq 0, \quad i = 1, \dots, N. \end{cases} \tag{5}$$

In this problem, we are looking for the portfolio composition $x = \left[x_1, \dots, x_N\right]^T$ for which the utility function is maximized. There are two constraints: the sum of the weights must be equal to one and the weights must be non-negative. The problem specified above can be understood as follows: by knowing parameter *k*, we can find the optimal portfolio composition. Note also that if:

- $k = 0$, the optimal portfolio is equal to the maximum return portfolio, that is, the portfolio will be composed of only one asset with the highest expected return,
- as $k \to \infty$, the optimal portfolio is approaching the minimum-variance portfolio and
- for different values of $k \in \left(0, \infty\right)$, we obtain different points of an efficient set.

In the above-mentioned problem, all the assets are assumed to be risky assets, that is, the variances of the returns are greater than zero. Investment in risk-free asset is not allowed. However, this restriction can be relaxed. We can generalize the optimization problem, allowing investment in risk-free asset. Note that the expected return of risk-free asset is equal to the risk-free rate r_{rf} and its variance is equal to zero, that is, the risk-free rate is not random. Then, the generalized optimization problem is as follows:

$$w = \begin{cases} \arg\max\limits_{x} \sum\limits_{i=1}^{N} x_i \cdot E\left(r_i\right) + x_{rf} \cdot r_{rf} - k \cdot \sum\limits_{i=1}^{N}\sum\limits_{j=1}^{N} x_i \cdot \sigma_{i,j} \cdot x_j \\ \sum\limits_{i=1}^{N} x_i + x_{rf} = 1 \\ x_{rf} \geq 0 \\ x_i \geq 0, \quad i = 1, \dots, N, \end{cases} \tag{6}$$

where x_{rf} is the weight of the risk-free asset and r_{rf} is the risk-free rate. The constraints are the same: the sum of all the weights (i.e. the sum of the weights of risky assets and the weight of risk-free asset must be equal to one) and all the weights must be non-negative.

3. Portfolio optimization backtesting

In the previous chapter, we proposed the optimization problems that can be applied to find the optimal portfolio. As already stated, the composition of such a portfolio is dependent on the investor's risk aversion, which is modelled by parameter *k*. However, the composition of the optimal portfolio also depends on the expected returns and covariances of individual assets' returns, which are, alas, not directly observable and must be estimated. Many methods and models exist to estimate future expected returns. In the further text, we will assume estimation from historical data assuming Gaussian i.i.d. returns. A more complex approach, which can be applied, is to estimate and predict the responses from ARMA-GARCH models. However, this approach is computationally complex and thus also involves high time requirements. The assumption of Gaussian i.i.d. returns, on the other hand, allows quick estimation.

Within the backtesting procedure, historical data are utilized. For each observation (day), we compute the portfolio composition based on the information set known at that moment; specifically, the weights of the portfolio at time *t* are determined based on the returns/prices of the assets in period (*t* – *m*, *t* – 1), where *m* determines the size of the past data that are utilized. To avoid look-ahead bias, it is vital to assure that the algorithm utilizes only information that would have been available at the time of the portfolio rebalancing.

Under such a set-up, we can compute the ex post portfolio returns $r_{P,t}$,

$$r_{P,t} = \sum_{i=1}^{N} r_{i,t} \cdot w_{i,t}, \tag{7}$$

where $r_{i,t}$ are the ex post observed returns and $w_{i,t}$ are the weights of assets in the portfolio, which are obtained by portfolio optimization based on the returns/prices of the assets in period $\left(t - m, t - 1\right)$. Then, we can also compute the ex post wealth (i.e. portfolio value) path,

$$W_{t+1} = W_t \cdot \left(1 + r_{P,t}\right), \tag{8}$$

which is vital for further analysis.

We set $W_0 = 1$, that is, at the beginning of the backtesting period, we start with the wealth of one currency unit. Then, we cycle through the backtesting period and after each new observation we re-optimize the weights according to (5) or (6), rebalance the portfolio and compute the ex post portfolio return according to (7) and the value of wealth according to (8). Clearly, we are interested in the value of final wealth, W_T, namely the wealth that we possess at the end of the backtesting period.

Regrettably, the final wealth should not be the only characteristic in which the investor is interested. As the investor is risk averse, he dislikes the risk. Here, we substitute the risk with the maximum drawdown of the wealth over the backtesting period. The term maximum drawdown is explained below.

If we assume wealth path $\{W_t\}_{t=0}^{T}$, at time $\tau \in (0, T)$, we can measure the decline from the historical maximum peak; this measure is called drawdown and can be computed as follows:

$$DD_\tau = 1 - \frac{W_\tau}{\max\limits_{t \in (0,\tau)} W_t}. \tag{9}$$

Note that (9) is stated as a percentage – that is, the size of the decline that we suffer at time τ relative to the previous maximum wealth (the highest peak). However, we can extend the ratio so that we measure the maximum drawdown over the period $(0, T)$:

$$MDD_{0,T} = \max_{\tau \in (0,T)} \left[DD_\tau \right]. \tag{10}$$

The maximum drawdown (MDD) is the worst decline in wealth over the selected period, meaning the maximum relative difference between the peak value and the subsequent valley value. For further explanation, see for example Chekhlov et al. (2005) or Magdon-Ismail et al. (2004), who studied the relationship between maximum drawdown and geometric Brownian motion.

The computations were performed in Matlab. Applied algorithms as well as a further description of the general backtesting framework are provided by Kresta (2015a, 2015b).

4. Empirical part

In the previous section, we described two portfolio optimization problems – with and without risk-free investment possibilities. As stated, the optimization problem allowing risk-free investments (6) is the generalization of the problem that does not allow risk-free investments (5) and thus it should both enlarge the portfolio composition possibilities and allow the formation of better portfolios in terms of the trade-off between expected return and variance. Further, we analyse whether the wealth path obtained by means of the backtesting procedure improves. Specifically, we focus on the final value of the wealth and the value of the maximum drawdown.

Data set

The data set utilized consisted solely of the stocks incorporated into one of the American stock market indices – the Dow Jones Industrial Average (henceforth DJIA). We assumed all the components of the index as of 6 October 2014, except the stocks of the Goldman Sachs Group, Inc. (Yahoo Finance ticker GS) and Visa Inc. (Yahoo Finance ticker V). These two stocks were excluded from the data set as we were not able to obtain historical data for a long enough period. Thus, the data set consists of only the 28 remaining stocks.

The historical data of the stocks included in the data set were obtained from the Yahoo Finance website[1] over the period from 3 January 1994 to 31 December 2014 (6,048 daily observations for each stock). However, we estimated the parameters from 250 observations; thus, the backtesting was performed in the period from 30 December 1994 to 31 December 2014, leaving the first year of data for initial parameter estimation.

4.1 Empirical results

The backtesting results (i.e. the ex post wealth paths) of the portfolio optimization problem without risk-free investment possibilities are shown in Figure 1.

From the figure, we can conclude that the lower the value of parameter k (the less risk averse the investor is), the higher the final wealth. It is also apparent that the higher the value of parameter k (the more risk averse the investor is), the lower the volatility and drawdowns. Moreover, we can see that, irrespective of the value of parameter k, all the wealth paths dropped significantly in the period 2008/2009 due to the subprime crisis and in 2010/2011 due to the country credit risk crisis. A steady decline also occurred in the period 2000–2004.

[1] http://finance.yahoo.com

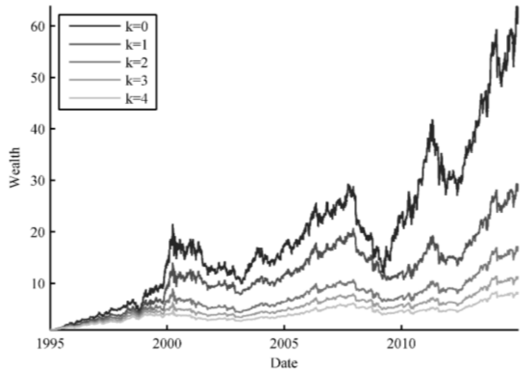

Figure 1 Wealth paths obtained by portfolio optimization not allowing risk-free investments

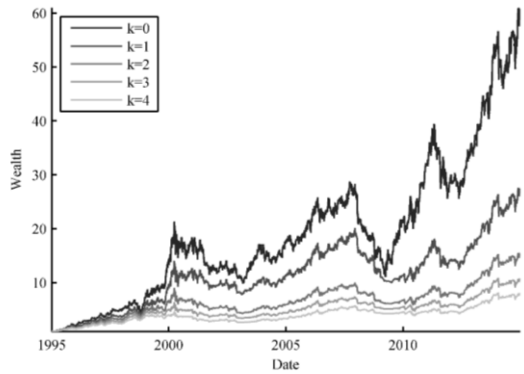

Figure 2 Wealth paths obtained by portfolio optimization allowing risk-free investments

The backtesting results (i.e. the ex post wealth paths) of the portfolio optimization problem that allows risk-free investments are shown in Figure 2. In the specification of the problem, we assumed the risk-free rate to be equal to 2% p.a. From the visual comparison of the two figures, we can conclude that the ex-post wealth paths obtained are identical for the two problems. The values of the final wealth and the maximum drawdown will be analysed more precisely in the text below.

It is also important to analyse the changes in portfolio composition over time. To be more specific, we expect that during periods of general decline in the financial markets, the weight of risk-free asset is increased and the weight of risky assets is decreased. The evolution of the relation between the weight of risk-free asset and the weight of risky assets is depicted in Figure 3. The figure shows the evolution of the weights obtained by means of backtesting the optimization problem allowing risk-free investments and assuming k to be equal to 4 and the risk-free rate to be equal to 2% p.a. As can be seen, in 2009, the weight of risk-free asset was especially greater; however, based on Figure 1, we can conclude that the increase occurred quite late, actually after the wealth had already dropped. The same situation applies to the period 2000–2004.

To compare the influence of the possibility of risk-free asset investments, we compared the values of final wealth and maximum drawdown. We analysed the portfolio optimization problem without risk-free investment possibilities (5) and with risk-free investment possibilities (6). In the latter optimization problem, we assumed the risk-free rates of 0%, 2%, 5% and 10% p.a. Note that the most realistic case is the risk-free rate of around 0%–2% p.a.; however, for the purposes of the analysis, we also assumed higher values.

The values of the final wealth are depicted in Table 1, while the values of the maximum drawdown are

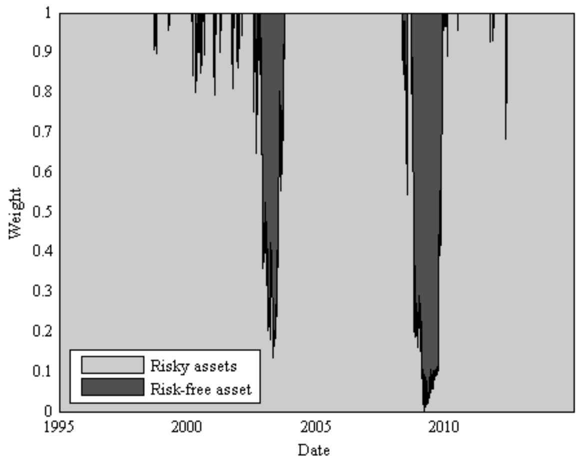

Figure 3 Evolution of portfolio compositions

summarized in Table 2. From the results, we observe that for realistic values of the risk-free return (0%–5% p.a.), the allowance of risk-free investments in the portfolio composition yields smaller final wealth while possessing generally the same or higher maximum drawdowns. Actually, the maximum drawdowns are smaller in some instances; however, the differences are very small. For the unrealistically high risk-free rate (10% p.a.) and k equal to zero or one, the situation is the same; only for k greater than 1 does allowing risk-free investments in the portfolio composition provide an improvement in both the final wealth and the maximum drawdown. However, the differences are rather small and we also have to keep in mind that the risk-free rate of 10% p.a. is unrealistically high.

There is another surprising relationship. We would expect that by increasing the risk-free rate, the observed final wealth would increase and the maximum drawdown decrease. However, the opposite is true for small values of parameter k (less risk-averse investors) and realistic risk-free rates.

Table 1 Final wealth under different models

Model	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
no risk-free asset	61.62	28.29	16.65	11.06	8.10
risk-free asset ($r_{rf} = 0\%$)	61.39	27.46	14.82	10.19	7.66
risk-free asset ($r_{rf} = 2\%$)	58.76	26.51	15.01	10.43	7.81
risk-free asset ($r_{rf} = 5\%$)	54.41	26.23	15.53	10.89	7.84
risk-free asset ($r_{rf} = 10\%$)	55.37	26.49	16.94	11.37	8.20

Table 2 Maximum drawdown under different models (in %)

Model	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
no risk-free asset	59.64	49.41	52.88	47.77	44.74
risk-free asset ($r_{rf} = 0\%$)	59.93	49.93	52.69	47.12	44.19
risk-free asset ($r_{rf} = 2\%$)	61.50	50.80	52.86	46.83	44.25
risk-free asset ($r_{rf} = 5\%$)	63.02	50.73	52.45	46.56	44.85
risk-free asset ($r_{rf} = 10\%$)	61.78	48.23	51.07	47.29	44.05

5. Discussion

According to the presented results, the allowance of risk-free investment makes no positive contribution to the final wealth or the maximum drawdown. However, the reason for these results lies in the way in which the future returns are estimated. In our paper, we assumed that the joint distribution of one-day-ahead returns can be estimated from the preceding 250 daily returns. In other words, the investor maximizes his/her utility function based on the 250 most recently observed daily returns. However, by applying this method of prediction, the strategy reacts to the market declines with a delay; that is, in the early stages of market declines, it is advisable to invest in risky assets, while at the end of market declines and during subsequent market recovery stages, investing in risk-free asset may be preferable; compare Figure 3 with Figure 1. Thus, in this way, it is impossible for the investor to predict whether there will be a general decline in the market and he/she should invest in risk-free asset or there will be growth and he/she should invest in risky assets. Thus, it is better for the investor always to invest in risky assets than to try to predict the future market movements (by the simple method based on historical observations) and move funds between risky assets and risk-free asset.

6. Conclusion

The problem of finding a proper portfolio composition is in the focus of both academics and practitioners. In the paper, we utilized the mean–variance framework introduced by Markowitz to analyse whether the performance of portfolio optimization will improve as investment in risk-free asset is allowed. The performance of portfolio optimization was measured by the value of final wealth as well as the value of maximum drawdown in the chosen backtesting period.

The results indicated that the introduction of risk-free asset ensures an improvement neither in the value of the final wealth nor in the maximum drawdown. Thus, under the assumption made, we conclude that the introduction of risk-free investment possibilities does not improve the investment opportunities. However, the assumption that we made, that is, that the returns are i.i.d. with Gaussian distribution, is very simplifying. We kept this assumption due to the computational complexity of other approaches.

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