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To Possibilities of Solution of Differential Equation of Logistic Function

Summary

The contents of the paper relates to articles of authors using the mathematics in economics with application in Excel. It focuses upon the information flow using the logistic function, which describes the spread of information in a system with limited capacity and the solution by Bernoulli's differential equation. The aim of the article is to analyse the application of the logistic function to describe the spread of information or number of customers in a hypermarket depending on time. It is expressed by Bernoulli's differential equation with constant coefficients. Theoretical starting points are being illustrated on practical examples using the approximate method for solving the integral by systematic integration. It presents the process, analysis and specification of the relative mistake. There is an example showing the programming of the solution of a primitive function in Excel, its process, discussion on results. Finally there is analysis of the solution of economic calculations in Excel, the procedure of task transformation and the comparison of obtained results with the reality. A given application may be used as an effective tool for modelling tasks on system saturation.

Keywords

logistic function, Bernoulli's differential equation, systematic integration, Excel

1. Introduction

The aim of logistic is to achieve the optimal solution of a certain topic in its application field. One of possible methods for solving it is the usage of logistic function. This function was introduced into the logistic by Pierre Francois Verhulst in the 19th century and it found its application in many fields.

Authors focused on the stream of information employing the logistic function which describes the spread of information in a system with limited capacity. They give solution by Bernoulli's differential equation. In addition to this authors present possibilities for its solution using a method of systematic integration and its application in Excel or similar spreadsheets.

2. Differential equation of logistic function

Logistic function $x(t)$ describing the spread of information or the number of customers in a hypermarket in dependence on the time t, in a system with limited capacity N has the following form

$$x(t) = \frac{a}{1 + b.e^{-c.t}} \quad a \neq 0 \tag{1}$$

where a, b, c are the parameters, t is the time variable.

By derivation of (1) with respect to time t we get

$$x'(t) = \frac{a.b.c.e^{-c.t}}{\left(1 + b.e^{-c.t}\right)^2} = \frac{-a^2.c/a + a.c + a.b.c.e^{-c.t}}{\left(1 + b.e^{-c.t}\right)^2} = a.c.x(t) - \frac{c}{a}x^2(t) \tag{2}$$

Let us suppose, that the velocity of spreading the quantity $x(t)$ of economic information among N participants is proportional to the product of entities sending and receiving the information thus it fits the following equation

$$x'(t) = k.[N - x(t)].x(t) = k.N.x(t) - k.x^2(t) \tag{3}$$

By comparing (2) and (3), we get after arrangement

$$a = N, \qquad c = k.N, \ b = [N - x(0)]/x(0),$$

where $x(0)$ is a number of entities sending the information in the time 0.

Expression (3) is Bernoulli's differential equation with constant coefficients $k.N$ and $-k$. Its solution is the function

$$x(t) = \frac{N}{1 + \dfrac{N - x(0)}{x(0)} \cdot e^{(-k.N.t)}} \qquad (4)$$

To evaluate the coefficient k it is necessary to take the information from the „interior" of the interval $(0, \infty)$.

Example 1. Spreading of the fame [1]. Let us suppose that there are 1200 students in the college area. 8 from those students start to spread certain fame. Further on we suppose that 18 hours later there were 800 students familiar with this fame.

How long does it take to spread the fame among more than 1100 students?

Solution. To solve this problem we use the function (4) with parameters $N = 1200$, $x(0) = 8$.

The parameter k is evaluated from the condition

$$x(18) = 800 = \frac{1200}{1 + \dfrac{1200 - 8}{8} \cdot e^{(-k.1200.18)}},$$

where $k = -0,00026375433$ \qquad (5)

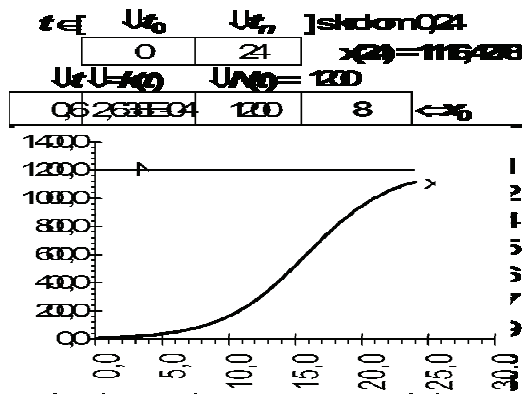The answer is on figure 1. It takes 24 hours to spread the fame among 1116 students.

Figure 1 Saturation of a system with a constant capacity

A general form of Bernoulli's differential equation is the following

$$x'(t) + f(t).x(t) = g(t).x^m(t), m \neq 0, m \neq 1, \qquad (6)$$
$f(t), g(t)$ are continous functions

Its solution for $m = 2$ is given by formula

$$\frac{1}{x(t)} = E(t)\left[\int_0^t \frac{f(s)}{E(s)} ds + C\right], \text{ where } E(s) = e^{\int_0^s g(u)du} \qquad (7)$$

The equation (3) is Bernoulli's differential equation with constant coefficients (with functions $f(t) = -k.N$, $g(t) = -k$) and with parameter $m = 2$.

If the functions f and g are not constant, (if for example, the capacity of N depends on time t) it is necessary to compute the integrals in (7). Generally, it is impossible to express the integral in square brackets in (7) in explicit form using elementary functions. Therefore it is necessary to choose the approximate method for its computation. The systematic integration represents one of possible methods.

3. Systematic integration

Interval $\langle a, b \rangle$ of the variable t will be divided on n equidistant parts with the step

$$h = (b - a)/n, \quad a = t_0, \quad b = t_n. \qquad (8)$$

Equidistant nodal points are:

$$t_i = t_0 + i.h, \quad i = 0, 1, \ldots n \qquad (9)$$

To deduce the formulas for systematic integration, we start with Lagrange's formula for the points $t_0, t_1, \ldots t_k$:

$$L_k(t) = \sum_{i=0}^{k} l_i(t).f(t_i), \quad l_i(t) = \frac{(t - t_0)(t - t_1)\ldots\overset{i}{\vee}\ldots(t - t_k)}{(t_i - t_0)(t_i - t_1)\ldots\overset{i}{\vee}\ldots(t_i - t_k)}$$
$$(10)$$

where in numerator and in denominator the i-th parenthesis is omitted [i. e. in the numerator the parenthesis $(t - t_i)$ and in the denominator the parenthesis $(t_i - t_i)$].

The polynomial $l_i(t)$ of $k - 1$ degrees is called the Lagrange interpolation polynomial with the property

$$l_i(t_j) = 0 \text{ for } i \neq j, \quad l_i(t_j) = 1 \text{ for } i = j,$$
$$i, j = 0, 1, \ldots k < n$$
$$(11)$$

The first three Lagrange interpolation polynomials for the points $t_0 = 0$, $t_4 = 4$, $h = 1$, $k = 4$ are presented on figure 2.
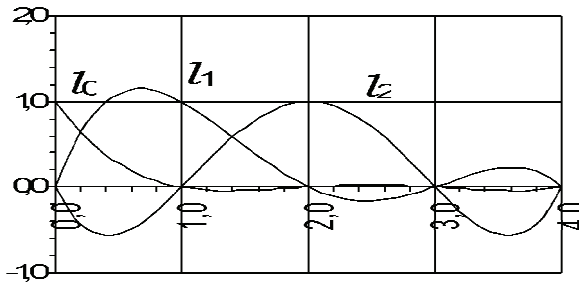
Figure 2 Lagrange interpolation polynomials of the 4th degree

Lagrange interpolation polynomial depends only on interpolation points. Lagrange's formula is a linear combination of Lagrange interpolation polynomials. The coefficients of this linear combination are the values of interpolated function in points (9). These points are called interpolation nodes or quadrature nodes. The Lagrange's formula is a theoretical basis for deducing the integration and compound quadrature formulas.

We will choose equidistant interpolation nodes (9) and computed quadrature formulas. These will have the following form

$$I_k = \sum_{i=0}^{k} H_i \cdot f(t_i) \text{, where } H_i = \int_a^b l_i(t)dt \qquad (12)$$

We shall consider that, the integral I_k is computed for various values of the upper bound b. These values are the interpolation nodes (9).

For $k = 5$ we get (after longer computation) the following formulas for systematic integration:

$$\int_a^{a+h} f(t)\,dt = \frac{h}{1440}\left[475f(t_0)+1427f(t_1)-798f(t_2)+482f(t_3)-173f(t_4)+27f(t_5)\right]$$

$$\int_a^{a+2h} f(t)\,dt = \frac{h}{90}\left[28f(t_0)+129f(t_1)+14f(t_2)+14f(t_3)-6f(t_4)+1f(t_5)\right]$$

$$\int_a^{a+3h} f(t)\,dt = \frac{3h}{160}\left[17f(t_0)+37f(t_1)+38f(t_2)+38f(t_3)-7f(t_4)+1f(t_5)\right]$$

$$\int_a^{a+4h} f(t)\,dt = \frac{4h}{90}\left[7f(t_0)+32f(t_1)+12f(t_2)+32f(t_3)+7f(t_4)+0f(t_5)\right]$$

$$\int_a^{a+5h} f(t)\,dt = \frac{5h}{288}\left[19f(t_0)+75f(t_1)+50f(t_2)+50f(t_3)+75f(t_4)+19f(t_5)\right]$$

$$(13)$$

Because it is valid

$$\int_{t_0}^{t_n} f(t)\,dt = \int_{t_0}^{t_0+5h} f(t)\,dt + \int_{t_0+5h}^{t_0+10h} f(t)\,dt + \int_{t_0+10h}^{t_0+15h} f(t)\,dt + \ . \ . \ + \int_{t_n-5h}^{t_n} f(t)\,dt$$

$$(14)$$

we get for $b = a + r.5h$ (where r is a natural number) a rule (algorithm) to compute the values of primitive function to the function $f(t)$ in

interpolating nodes (9). In practical applications we choose $r = 10$ until 20 and we get in majority of cases the results with a relative error of 5 % and less.

Example 2. Programming computation of values of primitive function

$$F(x) = C + \int_a^x f(t)\,dt \quad \text{ for } x \in \langle a, a + 10h \rangle \quad (15)$$

in Excel we proceed following way:

- Into the block C6:C16 we step by step insert the values $f(a), f(a+h), f(a+2h), \dots f(a+10h)$.
- Into the cell F3 we insert the step h.
- Into the cell F6 we insert integration constant C.
- Into the cells in block F7:F16 we gradually insert the instructions (16)

```
=F6+(475*C6+1427*C7−798*C8+482*C9−
   173*C10+27*C11)*$F$3/1440
=F6+(28*C6+129*C7+14*C8+14*C9−6*C10+C11)*$F$3/90
=F6+(17*C6+73*C7+38*C8+38*C9−
   7*C10+C11)*$F$3*3/160
=F6+(7*C6+32*C7+12*C8+32*C9+7*C10+0*C11)*$F$3*4/
   90
=F6+(19*C6+75*C7+50*C8+50*C9+75*C10+19*C11)*$F$3
   *5/288

=F11+(475*C11+1427*C12−798*C13+482*C14−
   173*C15+27*C16)*$F$3/1440
=F11+(28*C11+129*C12+14*C13+14*C14−
   6*C15+C16)*$F$3/90
=F11+(17*C11+73*C12+38*C13+38*C14−
   7*C15+C16)*$F$3*3/160
=F11+(7*C11+32*C12+12*C13+32*C14+7*C15+0*C16)*$F
   $3*4/90
=F11+(19*C11+75*C12+50*C13+50*C14+75*C15+19*C16)
   *$F$3*5/288
```

The values $F(a) = C, F(a+h), F(a+2h), \dots F(a+10h)$ appear in block F6:F16.

Note. The values of function $f(t)$ can be generated by a formula or, they can be observed values. If the number of nodes $n \neq 5r + 1$, the formulas for missing nodes are added by a less exact quadrature formulas, e. g. by the trapezoidal formula

```
=F16+(C16 + C17)*$F$3/2 for the 1st following
```
node or by the *Simpson's formula*
```
=F16+(C16 +4*C17+C18)*$F$3/3 for the 2nd
```
following node or by a *three-eights rule*
```
=F16+(C16+3*C17+3*C18+C19)*$F$3*3/8    for
```
the 3rd following node or by *Boole's rule*
```
=F16+(7*C16+32*C17+12*C18+32*C19+7*C20)
   *$F$3*4/90 for the 4th following node.
```

For the 5th following node we can use the last 5 formulas in (13) for the cells shifted down 5 rows.

4. Application in Excel

Excel is used mainly in economic computations. We will construct an application performing the formula (7) in Excel. Its construction in other spreadsheet program is similar. The application is based upon the systematic integration, which does not give the „names" of integrals in formula (7), but, for practice, it gives sufficiently accurate values in nodes (2). In addition to, it provides with a graph of the solution $x(t)$ in dependence on the time t. On screen it appears as on figure 3. Choosing $n = 100$ and sufficiently short interval [0, t_n] the results are accurate at least on 5 decimal places.

Figure 3 presents the computation of the example 1. graph $x(t)$ of the process of saturation is depicted on figure 1.
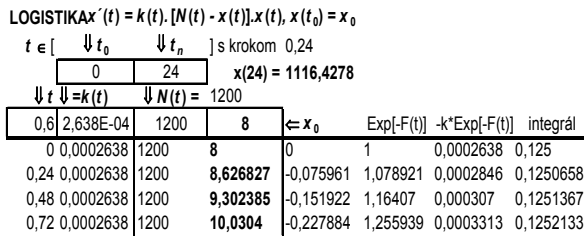
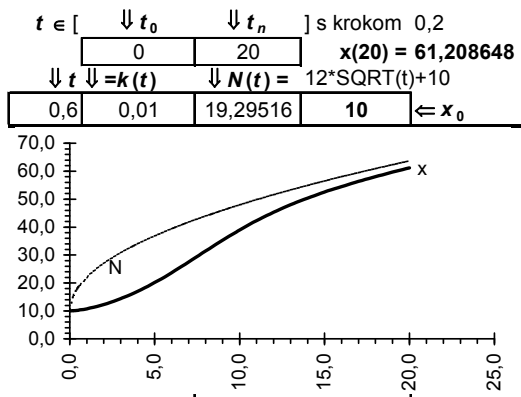**LOGISTIKA** $x'(t) = k(t). [N(t) - x(t)].x(t), x(t_0) = x_0$

$t \in [$	$\Downarrow t_0$	$\Downarrow t_n$	] s krokom 0,24			
	0	24	**x(24) = 1116,4278**			
$\Downarrow t$	$\Downarrow = k(t)$	$\Downarrow N(t) =$	1200			
0,6	2,638E-04	1200	**8**	$\Leftarrow x_0$	Exp[-F(t)]	-k*Exp[-F(t)]	integrál
0	0,0002638	1200	**8**	0	1	0,0002638	0,125
0,24	0,0002638	1200	**8,626827**	-0,075961	1,078921	0,0002846	0,1250658
0,48	0,0002638	1200	**9,302385**	-0,151922	1,16407	0,000307	0,1251367
0,72	0,0002638	1200	**10,0304**	-0,227884	1,255939	0,0003313	0,1252133

Figure 3   Application LOGISTIKA

$t \in [$	$\Downarrow t_0$	$\Downarrow t_n$	] s krokom 0,2
	0	20	**x(20) = 61,208648**
$\Downarrow t$	$\Downarrow = k(t)$	$\Downarrow N(t) =$	12*SQRT(t)+10
0,6	0,01	19,29516	**10** $\Leftarrow x_0$

Figure 4   Saturation of systems with increasing capacity

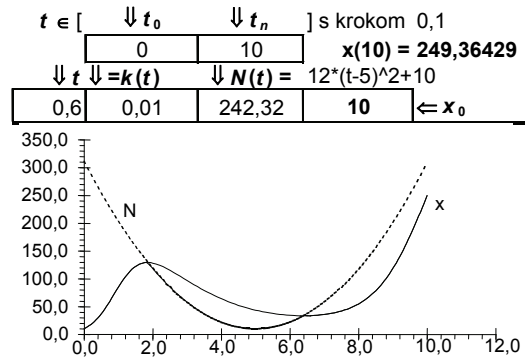| $t \in [$	$\Downarrow t_0$	$\Downarrow t_n$	] s krokom 0,1
	0	10	**x(10) = 249,36429**
$\Downarrow t$	$\Downarrow = k(t)$	$\Downarrow N(t) =$	12*(t-5)^2+10
0,6	0,01	242,32	**10** $\Leftarrow x_0$

Figure 5   Saturation of systems with variable capacity

If the function $n(t)$ is not decreasing, $x(t) < n(t)$ for all $t \geq 0$. see example 3 on figure 4. If the function $n(t)$ is decreasing, the $x(t) > n(t)$ for some $t \geq 0$. See the example 4 on figure 5. In that case is it convenient to place the number of those informed by the function

$$Y(T) = \text{MIN}(X(T), N(T)), \qquad (17)$$

because the number of informed entities cannot exceed the number of all participants in the system. This minimum is indicated by bold line on figure 5.

5. Conclusion

Economic problem is solved in such a way, that it is transformed into a corresponding mathematical model. For dynamic tasks it is mostly the common or partial differential equation. If the solution is known in explicit form, then the analysis is performed. The attributes of solution are compared with reality. According to this, one can consider how suitable it is to replace the real task by a mathematical model.

Ii is shown that, the differential equation (3) describes the saturation of a system well if the capacity of the N system is not decreasing. If the capacity of the system is decreasing than the correction is appropriate (17).

The given application can serve as an effective tool for modelling the task on system saturation.

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