Determining the product price of duopolist considering his limited offer and different demands of nodes

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Abstract. Game theory, and its specific area – spatial games, deal with the behavior of competitors. Spatial games focus on analyzing imperfect competition from a spatial point of view and the competitors represent companies operating in the market with the aim to attract customers and find the best location for their branch. Each company applies its own pricing policy, which affects its market share. In this article we present formulation and solution of a specific spatial game of two players who decide on the locations of their branches in space and want to maximize their revenues. The space is characterized by a graph, where location of customers and possible places of service represent its nodes. Customers choose one of the companies based on their total costs, consisting of the price of the product and shipping costs. The service in each of the nodes must be performed by either one or the other player. Such a situation can be analyzed using zero-sum games. The article presents the issue of determining the price of one player, based on a predetermined price of the opponent, to have player's revenues as high as possible. The game considers limited offer of the first player and different demands in each of the nodes.

Keywords: Spatial competition, Pricing, Matrix games, Imperfect competition, duopoly

JEL classification: *C 70, C 72, D 43*

1 Introduction

To operate in the market and maintain or improve its position, company must be competitive, which is closely linked to the prices of its products and services and their quality. To secure its position and market share, the company must proceed strategically. The concept of strategy can be found in various fields, one of which is game theory. It is a tool for analysing the strategic behaviour of players who can represent any entity in a conflict decision-making situation. In the market, it is precisely companies that find themselves in a conflict situation with competing companies offering the same or similar products and services. Each of these companies aims to gain as many customers as possible, increase their market share and maximize their profits. The path to success is defined at the outset by several factors. One of them is the choice of location, whether it is warehouses, branches, operations, production facilities, or equipment. Location models address this issue, while the problem of location can be defined at the level of municipalities, cities, regions, or states.

How a company behaves in the market is also influenced by the type of structure of the market in which it operates. In the market, we generally distinguish between perfect and imperfect competition, both of which are characterized by their specific features. Imperfect competition is characterized mainly by the ability to set and control product prices by companies or manufacturers (Samuelson and Nordhaus 2010).

The behaviour of the firms in the market of imperfect competition must consider the decisions of other subjects (whether on the supply or demand side). That means, when deciding on the quantity and price of the offered product, the company must, as part of an oligopolistic market structure, consider the steps of other companies. Such strategic interactions and their market specifications can be explained using game theory (Varian 1992). From the point of view of the game theory, the decision-making situation of individual oligopolists can then be considered as a game in which players try to maximize their expected payment by their strategic behaviour.

In the article we present the duopolistic market, defined by two companies operating in the market, in a space that can be characterized by a graph. Players make their decisions about the location of the operation simultaneously. It is a one-round game, the results of which are determined by setting the prices of their products, which also affect their respective market share (and thus sales). We will design an original mathematical model, based on which it is possible to set the price for one player based on the price of his opponent, which is known in advance, so that his sales are as high as possible. We consider additional condition of a limited supply and various demands in each possible location, which also represents location of customers choosing one of the duopolists based on their lower costs, which include both the price of the goods and transport costs.

2 Models of spatial competition

The basis of an open market economy is free competition, which is a conflict of interest The analysis of the oligopolistic market in space is currently increasingly discussed topic. One of the first to address this issue was the mathematician and economist H. Hotelling (1929), who presented a model based on the presence of two companies looking for the most advantageous position in the linear market. The model is the basis of many theories of product differentiation and location, but despite its applicability, it has undergone many criticisms. For example, C. D'Aspremont, J. Jaskold Gabszewicz and J.-F. Thisse (1979) points out its flaw and proves that it is not possible to have a balance if companies are close to each other. The result of their modified model is a model whose solution ensures the existence of equilibrium at any point in the market (D'Aspremont et al. 1979).

Even though the beginnings of the issue of spatial models are associated with Hotelling, in fact, the first known attempt to analyse economic activity in space is associated with the Thünen (1826), whose theory explained the location of production activities in an isolated city-state with land and homogeneous resources (Gehling, 1968). Weber (1909) later developed a theory of the location of industry.

Also, before Hotelling, in 1924, Fetter, one of the first authors to lay the foundations for the analysis of relationships and interdependencies between firms, published his work with a significant impact on network competition theory. Unlike Hotelling, Fetter focused on modelling demand behavior, not on optimal decisions (Biscaia and Mota, 2013). A further extension of Fetter's work can be found in the publications of many other authors, such as (Hamoudi and Martín-Bustamante, 2011) and (Hamoudi.a Risueño, 2012).

Other publications are proof that Hotelling's model has laid the foundations for several other works dealing with this issue. (Beath and Katsoulacos, 1991) is also based on his model. The authors deal, among other things, with the price competition of the spatial duopoly. Customers located along the linear market, forced to travel if they want to buy the products on offer, are the only ones who bear the transport costs. The location is an exogenous parameter for the companies, so price is their only decision variable (Beath & Katsoulacos, 1991).

3 Determining the product price of duopolist based on best response

In this section, we will present an original mathematical model that allows us to determine the price of the product of duopolist based on the determined price of the opponent in the case of a specific spatial game with the assumption of limited capacity of the duopolist determining price of his product and different demand in each possible location. The idea of the paper will be based on (Čičková and Holzerová, 2020). The paper was focused on modelling the pricing of the product price of duopolist based on best response model in case where each of the customers always made a purchase from the player, where the total costs associated with the purchase were lower.The specific model was based on zero-sum game model.

In the model, like Hotelling in his basic model, we apply the basic assumptions: product homogeneity (both companies on the market offer a very similar product), zero production costs of companies and consumer indifference (due to the choice of manufacturer with unlimited capacity). Basic model also assumes one unit consumption in each node. We leave this assumption in the extended version.

The idea of spatial game is based on (Lopez and Čičková 2018). We will assume following: Let $V = \{1,2,\dots n\}, n \in Z^+$ be the set of customers and let there be graph $G = (V, H)$ where V represents nodes of the graph and $H \subset VxV$ represents set of the edges $h_{ij} = (v_i, v_j)$ from node v_i to node v_j, while for each oriented edge h_{ij} there is assigned real number $o(h_{ij})$ referred to as a valuation or value h_{ij}. Spatial game was formulated in so-called full-valued graph $\bar{G} = (V, \bar{H})$ with the same set of nodes as graph G, where $\bar{H}$ is set of the edges between each pair of nodes v_i and v_j, while their valuation is equal to the minimum price between nodes v_i and v_j of the original graph, $i, j \in V$. It is often assumed that $\mathrm{o}(h_{ij}) = d_{ij}$ where d_{ij} represents the minimum distance (the shortest path lenght) between the nodes v_i and v_j, then the matrix $\mathbf{D}_{nxn} = \{d_{ij}\}$ is the matrix of the shortest distances between the nodes v_i and v_j.

We assume there are two companies (players) $P = \{1,2\}$, offering a homogeneous product (good or service) in unlimited quantities, and these companies can place their branches in just one of the nodes, i.e., in any element of the set $V = \{1,2, \dots n\}$, which are also the locations of customers. Although both players offer identical products in unlimited quantities, the price of the products may be different. Let p_1 be the price of the product of player 1 and p_2 the price of the product of player 2. Each customer makes a purchase from any company (service is always carried out, i.e., lost demand is not considered). When choosing a company, customers consider the total cost of purchasing the product, which consists of the price of the product and the cost of transportation to the selected company. Transport costs are expressed as t per unit distance. If player 1 places his store in the ith node ($i \in V$) and player 2 places his store in the jth node ($j \in V$), player 1 gets the customer from the kth node ($k \in V$) only if $t * d_{ki} + p^{(1)} < t * d_{kj} + p^{(2)}$, while $t * d_{ki} + p_1 = n_{ij}^{(1)}$ and $t * d_{ki} + p_2 = n_{ij}^{(2)}$ are elements of cost matrices of customers $\mathbf{N}^{(1)}$ and $\mathbf{N}^{(2)}$. Otherwise, the customer from the ith node is served by player 2. If $t * d_{ki} + p_1 = t * d_{kj} + p_2$, players share the demand equally.

The basic situation, represented by a *fixed price model*, is where the prices of both products are known in advance and based on the above assumptions. Thus, elements of the payment matrix of player 1 $\mathbf{A} = (a_{ij}), i, j \in V$, (where the element a_{ij} represents the number of served nodes of player 1 in the case if player 1 operates in the ith node and the opponent in the jth node), are explicitly calculated. The elements of matrix $\mathbf{A}$ are quantified based on the stated elements of cost matrices of consumers as follows:

$$a_{ij} = \begin{cases} a_{ij} + 1, & if\ n_{ij}^{(1)} < n_{ij}^{(2)} \\ a_{ij} + 0.5, & if\ n_{ij}^{(1)} = n_{ij}^{(2)} \end{cases}$$

Such matrix characterizes a given game with a constant sum (where the game constant is equal to the number of nodes of the graph *G*). Equilibrium strategies can then be determined in a standard way based on the *min-max* principle. If the use of this approach does not lead to an equilibrium strategy, equilibrium strategies can be determined based on linear programming problem.

When determining the best response to an opponent's price, the price of the goods of the second player p_2 is known in advance. The price of the goods of the first player p_1 is in this model variable and the player would like to set it in a way to maximize his revenues. It is obvious that under such assumptions the elements of the payment matrix of player 1 will depend on the value of p_1.

The relationship between the elements of the payment matrix and the price p_1 is:

$$a_{ij}(p_1) = \sum_{i \in V} \frac{sgn\left(t * d_{kj} + p_2 - (t * d_{ki} + p_1)\right) + 1}{2}$$

Now it is possible to express the pricing for player 1 by this mathematical model:

$$w * p_1 \to \max$$
$$a_{ij}(p_1) = \sum_{i \in V} \frac{sgn\left(t * d_{kj} + p_2 - (t * d_{ki} + p_1)\right) + 1}{2}$$
$$\sum_{i \in V} a_{ij} x_i \geq w, j \in V$$
$$\sum_{i \in V} x_i = 1$$

The problem is discontinuity of the Signum function here, but the function can be replaced by binary programming problem. The new model includes following sets and parameters:

- $n \in Z^+$ – number of nodes
- $V = \{1,2, \ldots n\}$ – set of all nodes
- $d_{ij} \geq 0, i, j \in V$ – shortest distance between nodes *i* and *j*
- $t > 0$– costs per unit distance
- $p^{(2)} > 0$ – price of opponent's (player 2) product
- *M* – big positive number
- $\varepsilon -$ small positive number.

Variables:

- $w \in \langle 0, n \rangle$ – number of served nodes
- $x_i \in \langle 0,1 \rangle, i \in V$ – *i*th mixed strategy of player
- $p^{(1)} > 0$ – price of product of player 1
- $a_{ij} \in \langle 0, n \rangle$, $i, j \in V$ – payment matrix of player 1
- $b_{kij}^{(1)} \in \{0,1\}; k, i, j \in V,$
- $b_{kij}^{(2)} \in \{0,1\}; k, i, j \in V,$
- $b_{kij} \in \langle -1,1 \rangle;\ k, i, j \in$ V.

This situation can be described by this mathematical model:

$$w * p^{(1)} \to \max \qquad (1)$$
$$t * d_{kj} + p^{(2)} - (t * d_{ki} + p^{(1)}) \leq M * b_{kij}^{(1)}; k, i, j \in V \qquad (2)$$
$$t * d_{kj} + p^{(2)} - \left(t * d_{ki} + p^{(1)}\right) \geq -M * b_{kij}^{(2)}; k, i, j \in V \qquad (3)$$

$$b_{kij}^{(1)} + b_{kij}^{(2)} \leq 1; k, i, j \in V \tag{4}$$

$$b_{kij} = b_{kij}^{(1)} - b_{kij}^{(2)}; k, i, j \in V \tag{5}$$

$$b_{kij}^{(1)} * (t * d_{kj} + p^{(2)} - (t * d_{ki} + p^{(1)})) \geq \varepsilon * b_{kij}^{(1)}; k, i, j \in V \tag{6}$$

$$b_{kij}^{(2)} * \left(t * d_{kj} + p^{(2)} - (t * d_{ki} + p^{(1)})\right) \leq -\varepsilon * b_{kij}^{(2)}; k, i, j \in V \tag{7}$$

$$a_{ij} = \frac{\sum_{k \in V}(b_{kij} + 1)}{2}; i, j \in V \tag{8}$$

$$w \leq \sum_{i \in V} a_{ij} * x_i \, ; i, j \in V \tag{9}$$

$$\sum_{i \in V} x_i = 1 \tag{10}$$

The objective function (1) represents the revenue function of player 1. Equations (2) to (8) are used to determine the payment matrix of player 1. Equations (9) and (10) make it possible to determine the equilibrium mixed strategy of player 1.

3.1 Best response model with limited capacity of duopolist and different demands of nodes

In the previous section, we considered unit demand of individual nodes of the graph. It is obvious that a player's interest in each node is generally conditioned by the "size" of the demand of a given node, while in terms of this criterion, some nodes are more interesting for the player than the others. The size of demand can be related, for example, to the number of inhabitants. We will also leave the assumption of an unlimited offer and assume the limited offer of players. Considering the limited demand of nodes, which is given by the vector $\mathbf{g} = (g_i), i \in V$, let us also consider constraints on supply side. We will mark the maximum offered quantity of goods for player 1 as k_1 and the maximum offered quantity of goods for player 2 as k_2. Consumer demand will then be distributed among the players based on the following rules: the consumer seeks to minimize his costs. However, if the player's capacity is not sufficient, he must, despite the increased costs, move to the opponent.

When solving such game, the total capacity on the supply side needs to be considered. If it is possible to satisfy the whole demand of the nodes, that means if $k_1 + k_2 \geq \sum_{i \in V} g_i$, it is possible to use a game with constant sum. If it is not possible to satisfy the whole demand of the nodes, that means if $k_1 + k_2 < \sum_{i \in V} g_i$, any node would be equally advantageous for both players ($\mathbf{A}_{nxn} = (k_1)$, $\mathbf{B}_{nxn} = (k_2)$).

In case the prices of duopolists are known in advance (case of the *fixed price model*) and it is possible to satisfy the whole demand of the nodes ($k_1 + k_2 \geq \sum_{i \in V} g_i$), the calculation of elements of payment matrix of player 1 (**A**) and payment matrix of player 2 (**B**) can be written in the form of the following procedure:

LET $V = \{1,2, \ldots n\}$, $\mathbf{D}_{nxn} = (d_{ij})$, t, p_1, p_2, $\mathbf{g}_n = (g_i)$, k_1, k_2
LOOP $(i, j \in V)$ **DO**

$n_{ij}^{(1)} = t * d_{ij} + p_1$;
$n_{ij}^{(2)} = t * d_{ij} + p_2$;
$a_{ij} = 0$;
$b_{ij} = 0$;
LOOP $(k, i, j \in V)$ **DO**
IF $n_{ki}^{(1)} < n_{kj}^{(2)}$ **DO** $a_{ij} = a_{ij} + g_k$;
ELSEIF $n_{ki}^{(1)} = n_{kj}^{(1)}$ **DO** $a_{ij} = a_{ij} + 0.5g_k, b_{ji} = a_{ji} + 0.5g_k$;
ELSEIF $n_{ki}^{(1)} > n_{kj}^{(2)}$ **DO** $b_{ji} = b_{ji} + g_k$;
ENDIF
LOOP $(i, j \in V)$ **DO**
IF $a_{ij} - k_1 > 0$ **DO** $a_{ij} = k_1$, $b_{ji} = b_{ji} + a_{ij} - k_1$;
ENDIF
IF $b_{ji} - k_2 > 0$ **DO** $b_{ji} = k_2$, $a_{ij} = a_{ij} + b_{ji} - k_2$;
ENDIF

The determination of the equilibrium price of player as the best response to the set price of the opponent is different now. Unlike the basic model, we will consider the size of the demand of individual nodes. Let the demand of the nodes be given by vector $\mathbf{g} = (g_i), i \in V$ and the equation (17) will be replaced by equation:

$$a_{ij} \leq \frac{\sum_{k \in V}(b_{kij} + 1)}{2} * g_k; i, j \in V \qquad (11)$$

We consider the case where duopolist knows the limit of his capacity, but he does not know the limit of his opponent (he considers it to be large enough to satisfy the whole demand).

Then the elements of matrix **A** must meet the constraints:

$$a_{ij} \leq k_1; i, j \in V \qquad (12)$$

These relations will ensure (together with equations (1)-(7), (9)-(11)) the setting of such values of matrix **A**, that also meet the capacity limit for player 1.

4 Numerical example

Further illustrative example is inspired by the administrative division of the Slovak Republic. Let the nodes of graph *G* represent potential regions - the so-called catchment areas for the construction of new branches of two companies operating in the market in the position of two strong players ($P = \{1,2\}$). By regions (catchment areas) we will understand the regions of the Slovak Republic, represented by individual regional cities: 1-Banská Bystrica, 2-Bratislava, 3-Košice, 4-Nitra, 5-Prešov, 6-Trenčín, 7-Trnava and 8- Žilina. These 8 cities therefore represent the nodes of the graph G, $V = \{1,2, \dots 8\}$.

The demand of individual nodes is equal to the number of customers of these nodes and is represented by the vector:

$$\mathbf{g}^{\boldsymbol{T}} = (112; 115; 122; 122; 119; 108; 88; 109)$$

The numbers are given in thousands and rounded. This means that, for example, in the first node (i.e., in the city of Banská Bystrica) there are currently 160 thousand potential customers, whom companies can get and sell their products to. In the second node (Bratislava) there are 33 thousand more of them, that means 115 thousand customers and in the third node (Košice) 189 thousand. This continues until the last, eighth node, which corresponds to the city of Žilina and where there are currently 165 thousand potential customers.

Each of these consumers make purchase from one of the two companies (players) being aware of the costs they must bear if they decide to buy from a selected company. Based on these costs, he decides who from to buy the goods. Although the companies offer homogeneous services, their prices are not the same. However, in addition to the price of the goods, the total costs of the customers also include transport costs. Those per kilometre of distance are represented by *t*. We will consider different unit transport costs. It is clear, that the value of these costs also represents the "weight" between the player's price and the distance to go to the place of service. The matrix $\mathbf{D} = d(i,j)$, $i,j \in V$ is also known, is representing the shortest distances between individual regional cities and has the following form:

$$\mathbf{D}=\begin{bmatrix} 0 & 207 & 213 & 119 & 248 & 142 & 165 & 89 \\ 207 & 0 & 420 & 88 & 419 & 125 & 47 & 198 \\ 213 & 420 & 0 & 332 & 35 & 329 & 378 & 256 \\ 119 & 88 & 332 & 0 & 361 & 85 & 46 & 140 \\ 248 & 419 & 35 & 361 & 0 & 294 & 372 & 221 \\ 142 & 125 & 329 & 85 & 294 & 0 & 78 & 73 \\ 165 & 47 & 378 & 46 & 372 & 78 & 0 & 151 \\ 89 & 198 & 256 & 140 & 221 & 73 & 151 & 0 \end{bmatrix}$$

Let us have a situation in which the pre-known unit price of the second player's product will be $p_2 = 100$. However, let the consumer know that his price cannot differ from the other player's price by more than 50%. The product price of the player must be within interval $\langle p_1^{(lo)}; p_1^{(up)} \rangle$ where $p_1^{(lo)} = 50$ and $p_1^{(up)} = 150$. Let the unit costs per kilometre be *t*=0.2 and limited capacity of player 1 be 600. That means that he knows he cannot serve more than 600 units of demands, but he does not know the capacity of the opponent (he considers it to be large enough).

Based on our model we obtain these results, solved by GAMS and its solver Couenne (this is a problem of mixed integer nonlinear programming (MINLP)). The calculated price of the first player's product at the known price $p_2 = 100$ is at level $p_1 = 99.999$, at which he achieves revenues of 52,172.095, while serving almost 522 customers. The solution also gives the final payment matrix **A**:

$$\mathbf{A}=\begin{bmatrix} 600 & 462 & 600 & 462 & 600 & 353 & 462 & 356 \\ 433 & 600 & 600 & 115 & 600 & 203 & 115 & 325 \\ 241 & 241 & 600 & 241 & 356 & 241 & 241 & 241 \\ 433 & 600 & 600 & 600 & 600 & 437 & 584 & 325 \\ 241 & 241 & 539 & 241 & 600 & 241 & 241 & 241 \\ 542 & 600 & 600 & 458 & 600 & 600 & 570 & 433 \\ 433 & 600 & 600 & 311 & 600 & 325 & 600 & 325 \\ 539 & 570 & 600 & 570 & 600 & 462 & 570 & 600 \end{bmatrix}$$

The accuracy of the calculation can be verified using a *fixed price model* (where prices of both players are given in advance).

Strategy of the first player is then given by the vector $\mathbf{x}^T = (0; 0; 0; 0.01968; 0; 0.43633; 0; 0.54399)$. Interpretation of mixed strategies (the probability of strategy selection) is generally difficult. If it was possible to change the place of service (for example, daily revenues, distribution of the number of employees or distribution of a divisible commodity), player 1 should perform 1.968% of service in node 4, 43.633% in node 6 and 54.399% in node 8.

5 Conclusion

Every company wants to be successful in operating in the market. To gain its aims, it needs to be competitive and make strategic decisions like choosing location or price of its products. Spatial competition models are generally discussed topic focused primarily on the analysis of location decisions of players who aim to maximize their revenues. Game theory tools, due to their competitive nature, can support the decision analysis.

In our article we presented specific situation of spatial game dealing with placement on the graph, in which two companies operating in the market make simultaneous decisions about locations of their branches. At the same time, one of the two companies also decide on the price of its products while the price of his opponent is known in advance. That means, the price is a variable in the model and the duopolist tries to set it in a way to maximize his revenues based on the best response. Offer of this duopolist is considered to be limited. The demand, located in the nodes and represented by indifferent customers, is divided among duopolists and we assumed it to be different in each node. The duopolists compete to attract their potential customers, who choose one of them based on their total costs, consisting of product price and shipping costs. To analyze this situation zero-sum games can be used. The situation we also presented on numerical example where nodes represented regional cities of Slovak Republic, while demands of these cities were related to the number of their inhabitants. For finding optimal solution we used GAMS and its solver Couenne.

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