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The Czech Treasury Yield Curve from 1999 to the Present*

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Abstract
I estimate the Czech Treasury yield curve at a daily frequency from 1999 to the present. I use the parsimonious yield curve model of Nelson and Siegel (1987), for which I suggest a parameter restriction that avoids abrupt changes in parameter estimates and thus allows for the economic interpretation of the model to hold. The estimation of the model parameters is based on market prices of Czech government bonds. The Nelson-Siegel model is shown to fit the Czech bond price data well without being over-parameterized. Thus, the model provides an accurate and consistent picture of the Czech Treasury yield curve evolution. The estimated parameters can be used to calculate spot rates and hence par rates, forward rates or the discount function for practically any maturity. To my knowledge, consistent time series of spot rates are not available for the Czech economy.

1. Introduction

The yield curve is a fundamental determinant of almost all asset prices. The yield curve also influences many economic decisions. The Ministry of Finance of the Czech Republic is by far the biggest issuer of bonds denominated in Czech koruna and thus the Treasury yield curve is a natural benchmark yield curve of the Czech economy. However, consistent yield curve estimates over a long time period are not available. I estimate, day by day, the Czech Treasury yield curve from the beginning of 1999 to the present. The yield curve can be expressed in terms of spot rates, par rates, forward rates, or the discount function.

I use the parametric model of Nelson and Siegel (1987) to infer the Treasury yield curve from government bond prices. The Nelson-Siegel model, which has only four parameters, enables us to estimate the yield curve, without being over-parameterized, when the number of observed bond prices is limited. Despite the parsimonious number of parameters, the Nelson-Siegel model fits the data very well and thus provides an accurate picture of the Czech Treasury curve.

Besides providing the daily time series of the Czech Treasury yield curve for past more than eleven years, I elaborate on estimation issues of the Nelson-Siegel model. I suggest a parameter restriction which avoids abrupt changes in parameter estimates and thus allows for the economic interpretation of the model to hold.

This paper is inspired by Gurkaynak, Sack, and Wright (2006), who use the Nelson-Siegel model and its extension introduced by Svensson (1995) to estimate the U.S. Treasury curve from 1961 to the present. Slavík (2001) was the first to use

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the Nelson-Siegel model to estimate the Czech Treasury yield curve from Czech government bond prices. Another example is Málek, Radová, and Štěrba (2006), who use the Svensson model to comment on the predictive ability of model-implied forward rates. The two authors estimate and assess the yield curve for only one and two particular dates, respectively. In contrast, I run the estimation routine for more than 2900 days and evaluate the estimation results from both the cross-section and time series perspectives.

This paper is organized as follows. Section 2 reviews fundamental concepts of interest rates. Section 3 presents the Nelson-Siegel modeling framework, including a detailed discussion of the estimation issues. Section 4 provides an overview of our data. Section 5 demonstrates the yield curve estimation and defines estimation error measures. Section 6 presents the estimation results over the eleven year period – quality of fit, the evolution of the Treasury yield curve, Treasury rates versus swap rates – and also mentions the impact of the financial market crisis. Section 7 concludes. Additionally, some more figures, which are not required for understanding the text, are presented in *the appendix on the web site* of this journal. The resulting data are posted as *the data appendix* to this paper.

2. Yield Curve Basics

This section reviews fundamental concepts and relations of interest rates that will be used in the subsequent discussion. More details can be found, for example, in Cairns (2004) or in Cipra (2000).

2.1 The Discount Function, Zero-coupon Bonds, and Spot Rates

The key element in asset pricing is the discount function, or the price of a zero-coupon bond. Let $\delta_t(\tau)$, the discount function, denote the price at time t of a zero-coupon bond that pays 1 Czech koruna at the maturity date $t+\tau$. We use τ to denote the time to maturity.

The continuously compounded spot interest rate, or spot yield or zero-coupon yield for a zero-coupon bond maturing τ periods ahead is related to the zero-coupon bond price by

$$r_t(\tau) = -\frac{\ln\left(\delta_t(\tau)\right)}{\tau} \tag{1}$$

and conversely the zero-coupon price, or the discount function, can be written in terms of the spot rate as

$$\delta_t(\tau) = e^{-r_t(\tau)\tau} \tag{2}$$

Although continuously compounded interest rates may be mathematically convenient, interest rates are often expressed on a coupon-equivalent basis, in which case the compounding is assumed to be annual instead of continuous. The discount function is then expressed as

$$\delta_t(\tau) = \frac{1}{\left(1 + r_t^{ce}(\tau)\right)^{\tau}} \tag{3}$$

where $r_t^{ce}(\tau)$ is the coupon-equivalent or annually compounded spot interest rate. One can easily derive the relation between the continuously compounded yield and the annually compounded yield:

$$r_t(\tau) = \ln\left(1 + r_t^{ce}(\tau)\right) \tag{4}$$

The yield curve, or the term structure of interest rates, at a given date t is unambiguously represented by a set of spot interest rates with different maturities.

2.2 Coupon Bonds, Yield to Maturity, Par Rates, and Bootstrapping

In practice, bonds are almost solely issued as coupon bonds. Given the discount function, we can price any coupon bond by summing the price of its individual cash flows. For example, the price $P_t(\tau_n)$ at time t of a τ_n-period coupon-bearing bond that pays a face value of 100 Czech koruna at the maturity date $t + \tau_n$ and has n coupon payments left, where each coupon payment has a nominal value of C Czech koruna (C=100c, where c is the coupon rate) and the last coupon payment occurs at the maturity date, is as follows:

$$P_t(\tau_n) = \sum_{i=1}^{n} C\delta_t(\tau_i) + 100\delta_t(\tau_n) \tag{5}$$

where $\delta_t(\tau_i), i = 1,\ldots,n$ are discount functions (zero-coupon bond prices) with maturities $\tau_1,\ldots,\tau_n$.

Bond prices can be quoted in two different forms. The bond price $P_t(\tau_n)$ in (5) is called the dirty price. The dirty price is the actual amount paid when buying the bond. The clean price is an artificial price which is, however, the price most often quoted in the markets. It is equal to the dirty price minus the accrued interest. The accrued interest is equal to the amount of the next coupon payment multiplied by the proportion of the elapsed period from the previous coupon payment or from the issue date in the case of the bond's first coupon payment. The clean price is used because it does not jump at the time the coupon payment is paid out (or a bond goes ex-coupon). In contrast, the dirty price jumps at the time the coupon payment is paid out (or a bond goes ex-coupon), which leads to a saw-tooth evolution of the dirty price. For coupon bonds, yields to maturity are often quoted on markets. The yield to maturity is the constant interest rate that discounts the bond's cash flows so that they are equal to the price of the bond. Hence, the annually compounded yield to maturity y_t for the coupon bond from equation (5) fulfills

$$P_t(\tau_n) = \sum_{i=1}^{n} \frac{C}{\left(1 + y_t\right)^{\tau_i}} + \frac{100}{\left(1 + y_t\right)^{\tau_n}} \tag{6}$$

For coupon bonds, yields to maturity are often quoted on markets. But the picture they provide is imprecise. First, the yield to maturity is a measure of a bond's implied internal rate of return if it is held to maturity. This measure implicitly assumes that all coupon payments are reinvested at this same internal rate of return. Second, assume that the prices of two bonds with the same cash flow dates but different coupon rates are set according to (5), i.e., using the spot rates; then these two bonds will have different yields to maturity. This ambiguity of the yield to maturity is called the coupon effect. For these reasons, yields to maturity should not be used to represent the yield curve. Instead, spot rates or par rates should be used.

The par interest rate, or par yield, is the coupon rate $c_t(\tau_n)$ at which a τ_n-period coupon bond would trade at par, i.e., at its face value. Hence, according to the pricing equation (5), it must satisfy

$$100 = 100c_t(\tau_n)\sum_{i=1}^{n}\delta_t(\tau_i) + 100\delta_t(\tau_n) \tag{7}$$

This implies that

$$c_t(\tau_n) = \frac{1-\delta_t(\tau_n)}{\sum_{i=1}^{n}\delta_t(\tau_i)} \tag{8}$$

An example of par rates, quoted on markets, is interest rate swaps. A swap is an agreement between two counterparties to exchange cash flows in the future. In interest rate swaps, one counterparty pays cash flow equal to interest at a predetermined fixed swap rate on a notional principal for a number of years. In return, it receives interest at a floating rate on the same notional principal for the same period of time.

Let us observe a set of swap rate quotes for maturities $\tau_1 = 1, \tau_2 = 2, \ldots, \tau_n = n$ years. These par rate quotes assume annual coupons. For example, $c_t(10)$ denotes the par rate quote of a 10-year interest rate swap that matures in exactly 10 years at date t+10 and has 10 coupon payments on dates t+1, t+2,…, t+10. In this rather special case, when we have quotes of coupon rates related to all coupon payment dates, we can determine the discount function from the par rates by manipulating (8):

$$\delta_t\left(\tau_{j+1}\right) = \frac{1-c_t\left(\tau_{j+1}\right)\sum_{i=1}^{j}\delta_t\left(\tau_j\right)}{1+c_t\left(\tau_{j+1}\right)}, \quad j = 1,\ldots,n-1 \tag{9}$$

and $\delta_t\left(\tau_1\right) = 1/\left(1+c_t\left(\tau_1\right)\right)$. This recursive procedure, which converts par rates to discount functions, is called bootstrapping. We can proceed further from (9) to determine the spot rates. In practice, we have to adjust (7) and (9) to take into account day-count conventions.

2.3 Duration, Convexity, and Convexity Bias

Duration is a central figure in fixed-income analysis. It is a weighted average of the times when the cash flows pay out, with weights equal to the cash flows discounted by the yield to maturity:

$$D = \frac{1}{P_t\left(\tau_n\right)}\left(\sum_{i=1}^{n}\frac{\tau_i C}{\left(1+y_t\right)^{\tau_i}} + \frac{\tau_n 100}{\left(1+y_t\right)^{\tau_n}}\right) \tag{10}$$

where $P_t(\tau_n)$ is the price of the τ_n-period coupon bond, y_t is the annually compounded yield to maturity defined by (6), and τ_i measures time to maturity in years. Note that a zero-coupon bond has duration equal to its time to maturity. For a coupon bond, the effective time that the bond holder must wait to receive the notional principal is always longer than the duration. Equation (10) implies that for a given maturity and given set of spot rates (yield curve), the higher the coupon, the shorter the duration. Market practitioners often work with the following duration definition:

$$D^M = \frac{D}{1+y_t} \tag{11}$$

which is referred to as modified duration. Modified duration is used as a sensitivity measure of the price change relative to the yield to maturity change.[1] In the context of spot rates, the yield to maturity change can be thought of as a parallel shift of the spot yield curve. The first-order Taylor expansion of the bond price (6) with respect to y_t results in

$$\frac{\Delta P(\tau_n)}{P(\tau_n)} \approx -D^M \Delta y_t \tag{12}$$

where $\Delta P(\tau_n) = P(\tau_n, y_t + \Delta y_t) - P(\tau_n, y_t)$ and $D^M = -\left(1/P(\tau_n)\right)\mathrm{d}P(\tau_n)/\mathrm{d}y_t$ is the modified duration. This first-order approximation, however, is accurate only for small changes in yield to maturity because the relation between price and yield to maturity is nonlinear.

Convexity captures this nonlinearity. The second-order Taylor expansion of the bond price with respect to y_t results in

$$\frac{\Delta P(\tau_n)}{P(\tau_n)} \approx -D^M \Delta y_t + \frac{1}{2} C (\Delta y_t)^2 \tag{13}$$

where $C = \left(1/P(\tau_n)\right)\mathrm{d}^2 P(\tau_n)/\mathrm{d}y_t^2$ is the convexity of the bond. Convexity implies that the capital loss from an increase in interest rates will be smaller than the capital gain from a decline in interest rates. In particular, long-period bonds exhibit very high convexity,[2] which tends to depress long-period interest rates. This impact of convexity is referred to as the convexity bias. The convexity bias can be one of the main reasons for the noticeable concave shape of the yield curve at long maturities.

2.4 Forward Rates

Finally, the yield curve can also be unambiguously expressed in terms of forward rates rather than spot rates or par rates. Forward interest rates or forward yields are the interest rates between times $t+\tau_1$ and $t+\tau_2$ in the future ($\tau_2 > \tau_1$)

[1] A price sensitivity measure is often the only purpose of duration in the area of interest rate derivatives. The original definition of duration as a weighted average of the times when the instrument's cash flows pay out is then meaningless.

[2] Convexity increases roughly as the square of duration.

implied by current spot rates. In other words we are fixing the interest rate between times $t+\tau_1$ and $t+\tau_2$ in advance at time *t*.

The continuously compounded forward rate $f_t(\tau_1,\tau_2)$ can be synthesized from spot rates by a simple no-arbitrage argument – see, for example, Cairns (2004):

$$f_t(\tau_1,\tau_2)=\frac{r_t(\tau_2)\tau_2-r_t(\tau_1)\tau_1}{\tau_2-\tau_1} \tag{14}$$

The instantaneous forward rate is defined as the limit

$$f_t(\tau)\equiv\lim_{\tau_2\to\tau_1} f_t(\tau_1,\tau_2) \tag{15}$$

where $\tau=\tau_1$ denotes, in this case, the time to settlement of the instantaneous interest rate. Thus, we can express the instantaneous forward rate as a derivative of the spot rate with respect to the time to maturity:

$$f_t(\tau)=\frac{\partial}{\partial\tau}\left(r_t(\tau)\tau\right)=r_t(\tau)+\tau\frac{\partial r_t(\tau)}{\partial\tau} \tag{16}$$

This relation tells us that forward rates are above spot rates if the yield curve is upward sloping and below them if it is downward sloping. Conversely, the spot interest rate can be expressed as the average of the instantaneous forward rates with settlements between the trade date *t* and the maturity date $t+\tau$:

$$r_t(\tau)=\frac{1}{\tau}\int_{u=t}^{t+\tau} f_t(u)\mathrm{d}u \tag{17}$$

2.5 Day-Count Conventions

Czech government bonds are issued as coupon bonds with coupon payments once a year. For calculating cash flow time periods, the 30E/360 day-count convention is used. The 30E/360 convention assumes that a year has 12 months of 30 days each. The time period τ between dates $D_1/M_1/Y_1$ and $D_2/M_2/Y_2$ (read *Day/Month/Year*) is then calculated as

$$\tau=\frac{360\left(Y_2-Y_1\right)+30\left(M_2-M_1\right)+\left(D_2-D_1\right)}{360} \tag{18}$$

If D_1 is equal to 31, it is changed to 30 before plugging into (18). If D_2 is equal to 31, it is changed to 30 before plugging into (18). The Prague interbank deposit market PRIBOR and Czech interest rate swaps use the Act/360 day-count convention. Under this convention, the time period τ is calculated as the actual number of days between the two dates divided by 360.

3. The Yield Curve Model

If the Ministry of Finance issued a full spectrum of zero-coupon bonds every day, then we could simply observe the yield curve on the market. However, this is not the case. In the Czech Republic, only coupon bonds are issued by the Ministry of Finance and the number of bonds issued is very limited; the maximum number of

Czech government bonds traded on the market at the same time is 16. Hence, we need a yield curve model to infer spot rates from prices of existing bonds. Models proposed for estimating spot rates and hence forward and par rates fit some function of time to maturity and model parameters to observed coupon-bond prices.

My main purpose in estimating the Czech Treasury yield curve is to provide a general and consistent picture of the evolution of interest rates. In this paper I rely on the Nelson-Siegel (Nelson and Siegel, 1987) and Svensson (Svensson, 1995) models, which I refer to as the Nelson-Siegel framework. These models are usually preferred when the primary goal of the yield curve estimation is to provide sufficiently smooth yield curves which consistently reflect the underlying macroeconomic conditions and the investor's risk preferences. These models are parsimonious in number of parameters but allow for sufficiently rich shapes of yield curves while largely ignoring variations resulting from anomalous bond prices. Thus, it is not surprising that the Nelson-Siegel framework has become very popular among central banks and macroeconomic researchers. The Bank for International Settlements (BIS, 2005) reports that nine out of thirteen central banks which report their yield curve estimates to the BIS use the Nelson-Siegel framework.

Spline-based methods are another prominent approach to yield curve estimation (see, for example, Waggoner, 1997). Spline-based methods allow one to fit coupon-bond prices very precisely and hence they are considered the first-choice method if one is interested in small pricing anomalies. On the other hand, spline-based yield curves may not be smooth enough and may oscillate considerably from one day to another. This is an unappealing property if one is concerned with time-consistent yield curve estimation which is not affected by pricing anomalies.

3.1 Nelson-Siegel Framework

To simplify the notation, I drop the time subscript t in the forward rates, spot rates, and parameters in following sections. Nelson and Siegel (1987) assume that the instantaneous forward curve is the solution to a second-order differential equation with two equal roots:

$$f(\tau) = \beta_0 + \beta_1 e^{-\lambda\tau} + \beta_2 \lambda\tau e^{-\lambda\tau} \tag{19}$$

where $\theta = (\beta_0, \beta_1, \beta_2, \lambda)$ is a vector of parameters. Parameter λ is restricted to be positive. The forward rate (19) is a three-component exponential function. The first component, β_0, is a constant to which the forward rate tends as the time to settlement tends to infinity. The second component, $\beta_1 e^{-\lambda\tau}$, is a monotonically decreasing (or increasing, if β_1 is negative) exponential term, and the third component, $\beta_2 \lambda\tau e^{-\lambda\tau}$, can generate a "hump".

To increase the flexibility and improve the data fit, Svensson (1995) extends the Nelson-Siegel model by adding another "hump" component and thus increasing the number of parameters to six:

$$f(\tau) = \beta_0 + \beta_1 e^{-\lambda\tau} + \beta_2 \lambda\tau e^{-\lambda\tau} + \beta_3 \gamma\tau e^{-\gamma\tau} \tag{20}$$

where $\theta = (\beta_0, \beta_1, \beta_2, \beta_3, \lambda, \gamma)$ is a vector of parameters. Parameters λ and γ are restricted to be positive.

3.2 The Spot Rate Curve

The spot rates of the Nelson-Siegel modeling framework are derived by integrating the forward rates (19) and (20) according to (17). For example, the corresponding Nelson-Siegel spot rate curve is

$$r(\tau) = \beta_0 + \beta_1 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} \right) + \beta_2 \left(\frac{1 - e^{-\lambda\tau}}{\lambda\tau} - e^{-\lambda\tau} \right) \tag{21}$$

Again, the spot rate (21) is a three-component exponential function. The first component, β_0, is a constant to which the spot rate tends as the time to maturity tends to infinity. The second component, $\beta_1 \left(1 - e^{-\lambda_1\tau}\right) / (\lambda_1\tau)$, is monotonically decreasing (or increasing, if β_1 is negative) and governs the slope of the yield curve. This component tends to 0 as the time to maturity tends to infinity and tends to β_1 as the time to maturity tends to 0. The third component, $\beta_2 \left(\left(1 - e^{-\lambda\tau}\right) / (\lambda\tau) - e^{-\lambda\tau} \right)$, starts at 0, increases (or decreases, if β_2 is negative), and then tends back to 0 as the time to maturity tends to infinity. Thus, this component can be viewed as one that can generate a "hump". The Svensson model adds another "hump" component, $\beta_3 \left(\left(1 - e^{-\gamma\tau}\right) / (\gamma\tau) - e^{-\gamma\tau} \right)$, to the Nelson-Siegel spot rate equation (21). This component allows us to fit a second "hump" of the spot rate curve.

The functional form of the Nelson-Siegel and Svensson model components pins down the limits of instantaneous forward and spot rates:

$$f(0) \equiv r(0) = \beta_0 + \beta_1 \quad \text{and} \quad f(\infty) \equiv r(\infty) = \beta_0 \tag{22}$$

This implies that β_0 must be restricted to be positive to avoid negative long--period or, more accurately, infinite-period rates.

I graphically demonstrate the model spot rate components for March 2, 2007 in the right-hand chart of *Figure 2*. The components meet their theoretical interpretation as level, slope, and "hump". The left-hand chart of *Figure 2* presents the corresponding spot, forward, and par rates (see Section 5.1).

Diebold and Li (2006) offer a dynamic interpretation of the Nelson-Siegel model. If we assume λ to be constant over time, we can interpret the time-varying $\beta_{0,t}$, $\beta_{1,t}$, and $\beta_{2,t}$ as latent factors, and 1, $\left(1 - e^{\lambda\tau}\right) / (\lambda\tau)$, and $\left(1 - e^{-\lambda\tau}\right) / (\lambda\tau) - e^{-\lambda\tau}$ are then the corresponding factor loadings. This interpretation is in line with the results of Principal Component Analysis (PCA). PCA applied to the yield curve shows that the first three main components (factors) usually explain over 99% of the variability in interest rates. The first three components are then called the level factor, slope factor, and curvature ("hump") factor. This result is consistent across different data sets and time periods. According to this interpretation, the Nelson-Siegel model can be considered a three-factor model and the Svensson model a four-factor model. For an example of PCA applied to the Czech swap curve, see Kladívko, Cícha, and Zimmermann (2007).

Note that Diebold and Li (2006) do not use the Nelson-Siegel model to estimate spot rates from coupon-bond prices, but rather model the dynamics of spot rates. Spot rates are the input data into Diebold and Li's (2006) Nelson-Siegel-based model. In this context, the Nelson-Siegel framework is becoming increasingly popular. See, for example, Diebold, Rudebusch, and Aruoba (2006) for a dynamic macro--finance Nelson-Siegel model, or Christensen, Diebold, and Rudebusch (2010) for an arbitrage-free Nelson-Siegel yield curve model.

3.3 Estimating the Yield Curve Model

In estimating the forward rate curve and hence the spot rate curve, I set up an objective function that minimizes the weighted sum of the squared deviations between the observed and the model-implied prices of coupon bonds:

$$\hat{\theta} = \arg\min_{\theta} \sum_{i=1}^{N} \left(\frac{P_i - \hat{P}_i}{P_i D_i^M} \right)^2 \tag{23}$$

where N is the number of observed bonds, P_i is the observed dirty price of the coupon bond, $\hat{P}_i$ is the model-implied price (estimated price) of the coupon bond, and $(1/P_i D_i^M)$ is an optimization weight. The model-implied price $\hat{P}_i$ is calculated by plugging the model spot rates (for example, given by (21) for the Nelson-Siegel model) into the discount function (2) and then summing the bond's cash flows according to (5). The time fraction of cash flows is computed according to (18). Thus, I infer continuously compounded spot rates under the 30E/360 day-count convention.

I set the optimization weights equal to the inverse of modified duration defined by (11) multiplied by the observed dirty price of the coupon bond. Using these weights, we utilize the relation (12) and approximately minimize the sum of the squared deviations between the observed and the model-implied yields to maturity. Modified versions of this first-order approximation are typically employed when estimating bond yields from bond prices. Different authors choose slightly different weights. For example, Gurkaynak et al. (2006) set weights equal to $1/D_i^M$. See BIS (2005) for other weight specifications.

It is possible to directly minimize the sum of the squared deviations between the observed and the model-implied yields to maturity. This results in virtually the same parameter estimates, but it is computationally inconvenient because calculating the yield to maturity involves numerical root-finding, which must be run in each iteration of the objective function minimization. Direct minimization of the squared yield-to-maturity errors takes approximately three to eight times longer for the Czech data set depending on the number of observed bonds.

The estimation based on fitting yields to maturity implies a roughly equal mismatch of predicted versus observed yields to maturity, irrespective of maturity. The other estimation strategy is to minimize the unweighted sum of the squared deviations between the observed and the model-implied prices. However, this strategy can easily end up with a large mismatch of predicted versus observed yields to maturity at the short end of the yield curve. This is because of the smaller duration of short-period bonds, which makes their yields to maturity more sensitive to price changes. The pros and cons of fitting yields to maturity versus fitting prices are discussed in Slavík (2001) and Svensson (1995). Since in this paper we are concerned with interest rates and not prices, I choose to minimize the squared deviation of the yields to maturity.

3.4 Objective Function Optimization

The objective function (23), or its "yield to maturity analogue", represents a nonlinear least squares problem, and since the estimated bond price $\hat{P}_i$ is calculated as the sum of the bond's cash flows, there is no possibility for a simplifying transformation such as the logarithmic transformation. The implementation of the Nelson-Siegel framework is known to suffer from the following problems (see, for example, Cairns and Pritchard, 2001; Gimeno and Nave, 2009; Gurkaynak et al., 2006):

1. The objective function has multiple local minima.
2. The optimization algorithms are sensitive to the initial parameter values, especially to the "more nonlinear" λ and γ. This is not surprising given the multiple local minima.
3. Different combinations of parameter values can produce an equally good fit to the observed data.
4. The estimated parameters can abruptly change in value from one day to another. Again, this is to be expected given the possibility of equally good fits for different combinations of parameter values.

I discuss the first two problems in this section and the remaining two problems in Section 3.5.

I rely on the Matlab *lsqnonlin* routine in my implementation. *Lsqnonlin* is a trust-region-reflective algorithm designed for solving nonlinear least-squares problems. The algorithm is developed in Coleman and Li (1996). It enables the setting of lower and upper bounds for the variables to be optimized, which is handy for Nelson-Siegel and Svensson model estimation.

I run a simple simulation exercise to design and test the optimization procedure. First, I estimate the Nelson-Siegel model parameters from the given data set of Czech government bond prices. Second, I price the real existing Czech government bonds using the spot rates implied by the estimated parameters. Thus, I create synthetic Czech government bond prices. Finally, I re-estimate the model parameters from the synthetic Czech government bond prices and compare the re-estimated parameters with the original ones. I use the weighted price approach, i.e., the objective function (23). I run this simulation exercise for each settlement date available, which implies testing the estimation algorithm for different yield curve shapes and for different numbers of bonds.

The simulation exercise confirms that the optimization algorithm *lsqnonlin* is most sensitive to the initial value of λ. However, once the true λ is identified, the *lsqnonlin* routine converges to the true values of β_0, β_1, and β_2 very robustly. The results of the simulation exercise suggest that *lsqnonlin* succeeds in finding the global minima. Based on the results of the simulation exercise, I create the following sets of initial parameters:

– Set the initial values of β_0 and β_1 according to the relation (22), in which $r(0)$ is replaced with the shortest-period yield to maturity observed and r (∞)is replaced with the longest-period yield to maturity observed.
– Create a grid of different initial values of λ. The grid does not need to be very dense; a 5-point grid is sufficient for our data set. The maximum value of the grid equals 15, which implies that the "hump" component achieves its maximum for the maturity of 0.12 years.
– Set the initial value of β_2 equal to 0. This assumes no "hump" component.

The parameter estimates follow from the set of initial parameters for which the objective function reaches its lowest value.

I also run the simulation exercise for the Svensson model and the results are qualitatively the same, i.e., once the true λ and γ are identified, *lsqnonlin* robustly converges to the true betas. The initial parameters for the Svensson model are created accordingly.

3.5 Catastrophic Jumps and Parameter Restrictions

As already noted in the previous section, parameter estimates of the Nelson--Siegel framework can abruptly change in value from one day to another. Cairns and Pritchard (2001) refer to such changes as catastrophic jumps. For example, β_0, which determines the infinite-period spot rate, jumps down from 5% to 0% one day and jumps back to 5% the next day. To fit the short end of the yield curve, β_1 must also jump, because $\beta_0 + \beta_1$ pins down the instantaneous spot rate.

Catastrophic jumps also appear in Gurkaynak et al. (2006), as we can see from their parameter estimates.[3] For example, for June 9, 2006 Gurkaynak et al. (2006) provide parameter estimates of the Svensson model such that the β_0 estimate is basically 0 and the second "hump" component fits the long-period rates. Gurkaynak et al. (2006) state (p. 2299) that large jumps can appear in the parameter estimates (problem #4 from the previous section) but that the changes in the predicted spot rates are quite muted (problem #3 from the previous section).

However, catastrophic jumps in the β_0 and β_1 estimates preclude the economic interpretation of the model in terms of level, slope and, "hump". It is not reasonable to assume that long-period rates abruptly change from one day to another. Note that forward rates are especially sensitive to catastrophic jumps (see, for example, Figure 8 in Gimeno and Nave, 2009). Forward rates may already be affected in the maturity range in which we have estimation data points. *Figure 1A* in Appendix on the web site of this journal illustrates the impact of catastrophic jumps on forward rates. Therefore, it is desirable to avoid catastrophic jumps.

The problem of catastrophic jumps is data driven. For some combination of data points, the β_0 estimate goes to zero and the "hump" component fits the long-

[3] Gurkaynak et al. (2006) regularly update their parameter estimates on the internet: http://federalreserve.gov/econresdata/researchdata.htm.

-period rates of the spot rate curve. The "hump" component loading of the spot rate curve, $\left(1-e^{-\lambda\tau}\right)/\left(\lambda\tau\right)-e^{-\lambda\tau}$, starts at 0, increases to a maximum value of approximately 0.30 and then decreases back to 0 as the time to maturity tends to infinity. The "hump" component loading is multiplied by parameter β_2, which scales its magnitude and determines its sign. The Svensson model is more vulnerable to catastrophic jumps because it has an additional "hump" component governed by parameters γ and β_3.

Catastrophic jumps in the β_0 estimates, and thus the β_1 estimates, are allowed when λ is relatively small and β_2 can take relatively large values. The parameter λ determines the speed of peaking of the "hump." Large values of λ generate a quickly peaking "hump" at short maturities, while small values of λ generate a slowly increasing "hump" which peaks at long maturities and thus may fit most of the shape of the spot rate curve. Also note that λ governs the steepness of the curve. Small values of λ produce slow decay, which better captures rather flat curves, while large values of λ produce fast decay, which is suitable for rather steep curves. See *Figure 2A* in Appendix on the web site of this journal for an illustration of the "hump" and slope component loadings for different λ values.

The remedy I suggest for catastrophic jumps in the β_0 estimates is to set a lower bound for the λ (and γ) values. For the given maturity, τ^{max}, of the longest-period bond used in the estimation, I suggest to set a lower bound λ^{min} for the λ values in such a way that the "hump" component evaluated at λ^{min} reaches its maximum at one half of the longest maturity, but at most at the 10-year maturity. Thus, λ^{min} fulfills[4]

$$\arg\max_{\tau}\left(\frac{1-\exp\left\{-\lambda^{min}\tau\right\}}{\lambda^{min}\tau}-\exp\left\{-\lambda^{min}\tau\right\}\right)=\min\left(\frac{1}{2}\tau^{max},10\right) \tag{24}$$

I have determined this lower bound empirically and it prevents, at least for our data set, catastrophic jumps in the β_0 and β_1 estimates.

When λ is restricted from below by (24) and β_0 is restricted to be positive, I refer to the model as the restricted Nelson-Siegel model, while if λ and β_0 are only restricted to be positive, I refer to the model as the unrestricted Nelson-Siegel model. I place no restrictions on β_1 and β_2.

Naturally, the restriction of the λ domain reduces the flexibility of the model and thus may worsen the quality of the fit. However, I document in Section 6.1 that for our data set the restricted Nelson-Siegel model does not perform much worse than the unrestricted Nelson-Siegel model.

An alternative to the λ restriction could be to impose a penalty for β_0 into the objective function. For example, β_0 could be linked (in the least squares sense)

[4] The lower bound, λ^{min}, is an implicit function of τ^{max} and must be solved numerically. For example, if the longest-period bond in the data set has a maturity of 5 years, λ^{min} is equal to 0.713, i.e., the "hump" component reaches its maximum at 2.5-year maturity. If the longest-period bond in the data set has a maturity of 30 years, λ^{min} is equal to 0.1793, i.e., the "hump" component reaches its maximum at 10-year maturity.

with the yield to maturity of the longest-period bond. This procedure may solve catastrophic jumps in the β_0 and β_1 estimates, but my simulation exercise reveals that imposing this penalty leads to bias in the parameter estimates.

Gimeno and Nave (2009) find stable parameter values and also reduce the risk of objective function convergence to local minima for the Nelson-Siegel and Svensson models using genetic algorithms. They ran their analysis on daily Spanish government bond data and also note catastrophic jumps when using traditional optimization methods. Note that that Gimeno and Nave's (2009) catastrophic jumps in the β_0 estimates do not appear as severe as the catastrophic jumps in the β_0 estimates in our data set. They set up a genetic algorithm and show that they eliminate the catastrophic jumps and improve the quality of the fit. Gimeno and Nave (2009) use the weighted-price approach with an objective function similar to (23), although using slightly different weights.

I set up the genetic algorithm implemented in the Matlab Global Optimization Toolbox according to the parametrization of Gimeno and Nave (2009) given on pp. 2241–2245 and estimate the Nelson-Siegel model for several settlement days (selecting different yield curve shapes and different numbers of bonds) for our data set. However, I do not obtain lower objective function values than by using the deterministic *lsqnonlin* algorithm run from the sets of initial parameters outlined in the previous section. Moreover, the genetic algorithm takes much longer (on average hundreds of times longer) and also the genetic algorithm is a stochastic algorithm, which means that on each run it may converge to different parameter estimates.

I note that the Matlab *fminsearch* algorithm, which Gimeno and Nave (2009) use as a benchmark algorithm (p. 2246), is an implementation of the direct search method that does not use gradients and is not designed for nonlinear least squares problems. According to my simulation exercise, the *lsqnonlin* routine would be a more suitable benchmark algorithm.

4. Data Set

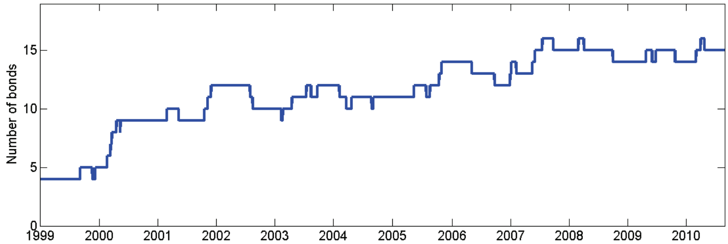
4.1 Bond Price Data

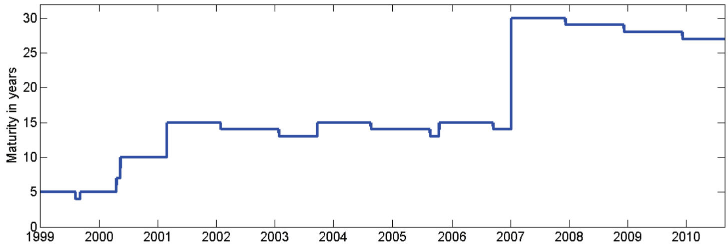
I use Czech government bond prices collected by the Prague Stock Exchange (PSE). For every outstanding bond, the PSE averages the end-of-day price quotes delivered by the Czech government bond market makers on each business day.[5] The PSE started to do so on July 14, 1997. In this paper, I present results based on the average of bid and ask prices, i.e., on mid prices.

While the PSE data set starts in mid-1997, I run the yield curve estimation from January 4, 1999 (the first business day in 1999). The reason is the insufficient number of bonds available before 1999. I am forced to exclude all bonds issued before January 1, 1998 because they were issued under a different taxation policy. The taxation policy influences the bond price and thus bonds with different taxation policies cannot be put together into one yield curve. At least four bonds should be used to identify the four parameters of the Nelson-Siegel model and four bonds are available shortly before the end of 1998.

[5] The rules – entitled "Determination of an Average Reference Price for a Bond" – are, at the time of writing this paper, downloadable from: ftp://ftp.pse.cz/Info.bas/Eng/Rules/bond_price.pdf.

Figure 1 The upper chart displays the number of bonds used in the estimation. The lower chart displays the maximum maturity of the bonds.





Czech government bonds are issued through primary auctions by the Ministry of Finance of the Czech Republic. Since 2000, each Czech government bond issue has been re-opened several times. The re-openings (tranches) make it possible to reach a fairly large total amount of each bond issue. A larger total amount issued should support the market liquidity of the bond. Starting January 4, 1999, I include in the estimation all government bonds with the following exceptions:

– I exclude all bonds with less than 180 days to maturity, since their price quotations often cannot be considered to represent real prices.
– I exclude all bonds before they reach 30 days after the issue date. This rule concerns only the first tranche for the given bond. Again, in some cases the price quotes of new bond issues behave oddly for the first few weeks.
– I exclude the 6.08%/2001 bond (issue number 27), since this bond appears to be constantly overpriced. This becomes obvious by just visually checking the yield to maturity curves – the yield to maturity of this bond is visibly too low. When used in the estimation, its observed yield lies as much as 60 basis points below the fitted yield and the difference between the observed and fitted yield stays negative for the whole life of the bond considered in the estimation. I am not able to provide an explanation for the odd behavior of this bond.
– I exclude the 4.85%/2057 bond (issue number 54). This 50-year bond is a low-volume issue which is not actively traded on the market.
– I also exclude floating interest rate bonds, since their use in yield curve estimation is not straightforward.

All the bonds used in the estimation are listed in *Table 3* in *Appendix A*.

In my opinion, the PSE data set is created carefully with a relatively small number of errors. I am forced to completely exclude four business days from the data set. For August 13, 2002, January 25, 2007, October 14, 2008, and September 9, 2008 too many bond price quotes are missing. In the rest of the PSE data set which is needed for the estimation, I have found 122 bond price quotes missing. Fortunately, the missing observations are just one-day (112 cases) or two-day (5 cases) price quote skips. I replace the missing price quote with its value from the preceding day. I have checked the PSE price quotes with available Reuters and Bloomberg data sets and found no substantial differences. Note that neither Reuters nor Bloomberg possesses complete bond price data over such a long period of time.

In total, I use 33,517 bond price observations spanning a period of 2,922 business days. In *Figure 1*, I plot the number of bonds used in the estimation and the corresponding maximum time to maturity of the bonds.

4.2 Short End of the Yield Curve

It is important to fit the short end of the Nelson-Siegel parametric yield curve. Otherwise, in some cases the short end can end up with unreasonable values, such as negative rates or extremely high rates. We have excluded all bonds with less than 6 months to maturity. Moreover, for 638 days of our data set, the shortest-period bond has more than one year to maturity. Thus, we have a lack of data points at the short end of the yield curve. Even if the estimated short-period interest rates are not to be used by the model users, they are employed in the estimation routine to discount coupon cash flows. Thus, it is important to keep their values in a reasonable range.

I use the arithmetic average of PRIBOR and PRIBID rates, which I refer to as PRIBOR MID. I use maturities of one week (1W), two weeks (2W), one month (1M), and three months (3M). I transform the PRIBOR MID rates into PRIBOR zero-coupon bond prices to be able to use them in the objective function (23). But the PRIBOR market is a different market than the government bonds market. This has become plainly evident during the recent financial turbulence. The PRIBOR market has a demanded a risk (credit) premium relative to government debt securities. Therefore, I underweight PRIBOR zero-coupon bonds in the objective function (23). I set their modified durations to be equal to the smallest modified duration among the government bonds. For our data, the smallest modified duration of government bonds implies the shortest maturity. Thus, the longer the maturity of the shortest-period bond, the smaller weight is attached to PRIBOR zero-coupon bonds in the objective function. The result is that the PRIBOR MID rates are not fitted as precisely and the short end of yield curve is reasonable. Additionally, using four more data points increases the degrees of freedom of the model.

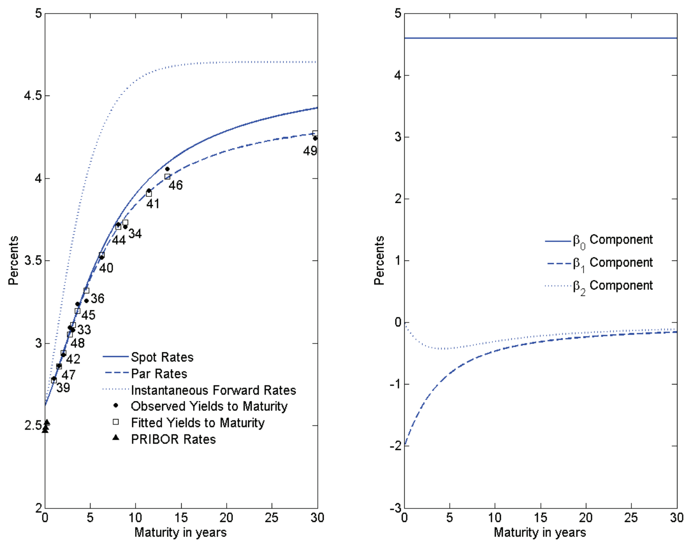
Of course, it could be more appropriate to use, for example, Czech Treasury bill rates for the short end of the yield curve. Unfortunately, this data is not available on a daily basis. PRIBOR rates are probably the only credible and consistent source of Czech short-maturity interest rates available on a daily basis.

5. Estimation Example and Error Measures

5.1 Example of Yield Curve Fitting

In the left-hand chart of *Figure 2*, I present spot, instantaneous forward, and par rates estimated with the Nelson-Siegel model on March 2, 2007. I have chosen

Figure 2 **Nelson-Siegel model for March 2, 2007. The right-hand chart plots the components of the Nelson-Siegel model. The left-hand chart captures the estimated spot, par, and forward rates (annually compounded). The model is estimated using the objective function (23). The numbers displayed under the yield to maturity marks are the issue numbers of the bond. The issue number is a unique identifier which is incremented by one for every new Czech government bond issue. See *Table 3* in *Appendix A* for a list of the bonds used in the estimation. The triangle marks denote the observed PRIBOR MID rates. The 1W, 2W, 1M, and 3M PRIBOR MID rates were employed in the model estimation.**



this settlement date arbitrarily from the time period between the issue of the 30-year bond (issue number 49) in 2007 and the start of the financial crisis in 2008.

The inferred rates are transformed, using relation (4), to annual compounding, which is usually preferred by market practitioners. There were 13 government bonds available on March 2, 2007 with a maximum maturity of 29.7 years and a minimum maturity of 1 year.

We can visually check the fit by comparing the observed (dots) and fitted (boxes) yields to maturity. The difference between the observed yield to maturity and the fitted (model-implied) yield to maturity is the estimation error. In other words, the error is the residual between the observed and fitted value. The observed 1W, 2W, 1M, and 3M PRIBOR MID rates are displayed as triangle marks. We can see from the left-hand chart of *Figure 2* that the Nelson-Siegel yield curve model does a good job of fitting the entire cross-section of yields to the maturity of government coupon bonds. The fit is the worst for the 6.55%/11 bond (issue number 36), which appears to be overpriced relative to the neighboring bonds.

5.2 Error Measures

The estimation error can be also expressed as the difference between the observed and fitted (model-implied) prices. The estimation error may indicate that

Table 1 This table reports the Root Mean Squared Error (RMSE) and the Maximum Absolute Error (MaxAE) for both the price and yield to maturity errors. The number in parentheses next to the MaxAE value is the corresponding issue number of the bond. Yield to maturity errors are measured in basis points. Price errors are measured in Czech hellers for a 100 Czech koruna notional principal. One heller is one hundredth of a Czech koruna.

	Yield to Maturity		Price	
Settlement	RMSE	MaxAE	RMSE	MaxAE
2 March 2007	3.2	5.9 (36)	24	51 (49)

the bond is mispriced. Furthermore, the error represents idiosyncratic noise, which is not captured by the model. Traditionally, idiosyncratic noise is assigned to liquidity issues, non-synchronous quotes, or data errors for different bonds. I suspect that the largest portion of the idiosyncratic noise in our data set is due to liquidity issues, which may be the reason for the bond prices being less connected to one another. This becomes pronounced at the beginning of the financial crisis in 2008. I revisit this issue in Sections 6.1 and 6.2.

To summarize the estimation error, I report the Root Mean Squared Error (RMSE) and the Maximum Absolute Error (MaxAE). The RMSE and the MaxAE for the yields to maturity are calculated as follows:

$$\text{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(y_i - \hat{y}_i\right)^2}$$

$$\text{MaxAE} = \max_i \left\{ | y_i - \hat{y}_i | \right\}, \quad i = 1, \ldots, n$$

where n is the number of government bonds for a given settlement date, y_i is the observed yield to maturity, and $\hat{y}_i$ is the fitted yield to maturity. Price errors are calculated by replacing the yield to maturity values with price values. Note that I do not include the PRIBOR rates in the estimation error measures.

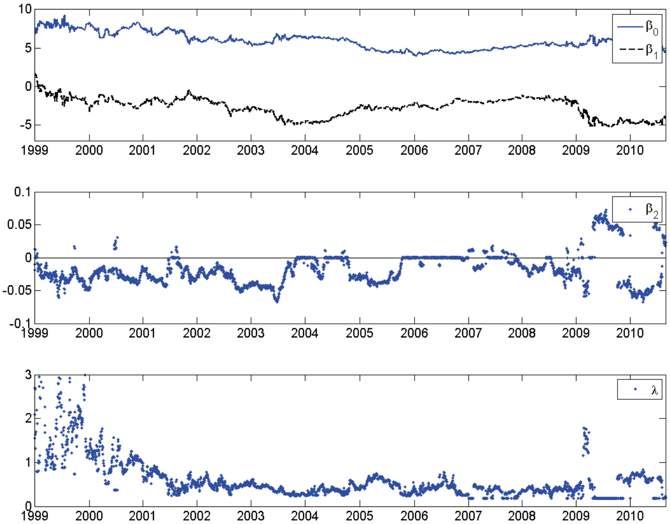
The error measures for March 2, 2007 are reported in *Table 1*, which confirms the plausible fit apparent from the left-hand chart of *Figure 2*. The worst yield to maturity fit, i.e., the yield to maturity MaxAE, of 5.9 basis points is for the 6.55%/11 bond (issue number 36), while the worst price fit, i.e., the price MaxAE, of 51 hellers[6] is for the 4.20%/36 bond (issue number 49). This discrepancy is due to the nonlinear relation between yields to maturity and prices as discussed in Section 2.3.

6. Estimating the Czech Treasury Yield Curve from 1999 to the Present

I run, day by day, the cross-section estimation of the yield curve described in Section 3 from January 4, 1999 to August 24, 2010. I use the restricted Nelson-Siegel model. The evolution of the parameter estimates is captured in *Figure 3*. There are no catastrophic jumps in the β_0 and β_1 estimates. However, on some days, especially at the beginning of our data set and during the last two years, the β_2 and λ estimates abruptly change in value. I present the parameter estimates of the unrestricted Nelson-Siegel model for which catastrophic jumps in β_0 and β_1 appear, in *Figure 3A* in

[6] One heller is one hundredth of a Czech koruna.

Figure 3 Evolution of the parameter estimates of the restricted Nelson-Siegel model. The β_0, β_1, and β_2 parameters are in percent. The parameters are estimated as discussed in Section 3.



Appendix on the web site of this journal. *Figure 4A* in Appendix on the web site of this journal compares the evolution of the β_0 estimates and the $\beta_0 + \beta_1$ estimates of the restricted Nelson-Siegel model with the yields to maturity of the longest-period bond and the shortest-period bond, respectively. *Figure 5A* in Appendix on the web site of this journal captures the same as *Figure 4A* but for the unrestricted Nelson--Siegel model.

Since March 2000, we have six bonds available and thus it is possible to identify the six-parameter Svensson model. But the Svensson model appears to be over--parameterized for our data set. The estimation of the Svensson model often converges to very close values of λ and γ. This implies multicollinearity and means that the second "hump" component of the Svensson model is redundant. As suggested in Svensson (1995), it is possible to impose a penalty for the λ and γ parameters into the objective function. Another possibility would be to simply restrict the domain of λ and γ values so that they do not overlap. Neither of these two restrictions is actually meaningful for our data set – the Svensson model only slightly outperforms the Nelson-Siegel model in terms of the error measures.

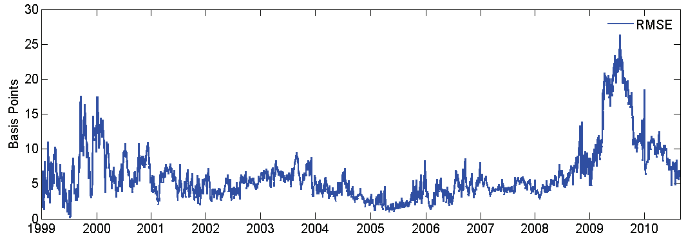
6.1 Error Measures Evolution

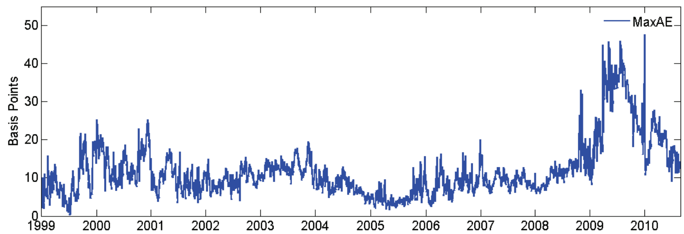
I illustrate the day by day evolution of the Root Mean Squared Error (RMSE) and the Maximum Absolute Error (MaxAE) for yields to maturity in *Figure 4*.

Figure 4 shows that the fit improves at the beginning of 2001 and stays generally very good until the end of the third quarter of 2008. The fit worsens during the financial market crisis, which started in 2008.

The longitudinal statistics of the error measures are relatively low. The average MaxAE is just 11.6 basis points. The maximum MaxAE, i.e., the maximum

Figure 4 The top chart displays the Root Mean Squared Error (RMSE) for yields to maturity. The bottom chart displays the Maximum Absolute Error (MaxAE) for yields to maturity. Both errors are measured in basis points.





absolute value of the residual between the observed and the fitted yield to maturity, is 47.6 basis points for the 4.10%/11 bond (issue number 54) on December 30, 2009. The yield curve estimate for December 30, 2009 is provided in *Figure 6A* in Appendix on the web site of this journal. On this day, the yields to maturity of the two neighboring bonds, at the short end of the yield curve, move in a different direction – the 4.10%/11 bond jumps up more than 30 basis points, whereas the 2.55%/10 bond drops more than 30 basis point. The yields to maturity of both bonds revert back close to their December 29, 2009 values during the first two business days of 2010. The average RMSE is only 6.2 basis points. The RMSE reaches its maximum of 26.4 basis points on July 21, 2009. The yield curve estimate for July 21, 2009 is provided in *Figure 7A* in Appendix on the web site of this journal. On this day, the yields to maturity are considerably divorced one from another. This inconsistent pricing of bonds is a typical picture of the Czech government bond market during the financial market crisis.

The longitudinal statistics of the error measures for both the restricted and the unrestricted Nelson-Siegel model and also for the unrestricted Svensson model are reported in *Table 2.* Imposing the lower bound (24) on the Nelson-Siegel model worsens the average RMSE only by about 0.3 basis points and it actually reduces the maximum MaxAE. The unrestricted Svensson model, which is frequently over-parameterized in our data set, does only about 1.2 basis points better in terms of the average RMSE compared to the unrestricted Nelson-Siegel model. The evolution of the error measures follows the same pattern for all three models compared. Imposing restrictions on the Svensson model, of course, brings its error measures even closer to the Nelson-Siegel model.

Table 2 This table reports the longitudinal error measures for both the restricted and the unrestricted Nelson-Siegel model and the unrestricted Svensson model (λ, γ, $\beta_0 > 0$). The error measures are for yields to maturity and are measured in basis points. Note that the Nelson-Siegel model has an estimation period about 14 months longer than the Svensson model. Therefore, the reported Avg RMSE for the Nelson-Siegel model must be reduced by about 0.1 basis point to be comparable with the Avg RMSE of the Svensson model.

	Avg RMSE	Max RMSE	Avg MaxAE	Max MaxAE
restricted Nelson-Siegel	6.2	26.4	11.6	47.6
unrestricted Nelson-Siegel	5.9	26.0	11.5	55.3
unrestricted Svensson	4.6	22.3	9.3	46.8

6.2 Impact of the Financial Crisis

The Czech bond market had already become nervous at the end of the first quarter of 2008, as documented by the spread of bond price quotes shown in the top chart of *Figure 5*. This chart displays the daily average price spread between the ask and bid prices.[7]

The average spread had been very stable around 30 hellers (standard deviation = = 1.3 hellers) for a 100 koruna notional principal from the beginning of our sampling period until March 2008, when it jumped to 42 hellers and oscillated around 45 hellers (standard deviation = 1.8 hellers) for the next six months. On October 8, 2008 the average price spread soared to 2.29 koruna and stayed very high, ranging from 1.04 koruna to 2.52 koruna (standard deviation = 32.1 hellers) until the end of 2009. From the beginning of 2010 to the last day of our data set the average price spread oscillated around 82 hellers.

The large difference between the ask and bid price quotes indicates the uncertainty of the market and its unwillingness to trade, which turns into inconsistent pricing of bonds. The linear regression of the RMSE (basis points) on the average price spread (hellers) over our entire data set has an R^2 of 42% and a slope coefficient of 0.067 (p-value <0.001). The relation between the RMSE and the average price spread is significant on different subsamples of the data set. We can expect the error measures to decline and the price spread to decrease to a reasonable level when the bond market starts to be fully functional. Also note that the λ and β_2 estimates suffer from catastrophic jumps during the financial crisis years. Again, we can expect that the λ and β_2 estimates will ultimately stabilize. The parsimonious Nelson-Siegel framework appears to be particularly suitable for the yield curve fitting during the financial crisis, when the pricing anomalies across the bonds are substantial. As already noted, the Nelson-Siegel framework does not fit each data point exactly, but rather smoothes the data points with a yield curve which reflects the underlying economic situation.

The price spread is clearly related to market liquidity. Traded values could be another measure of market liquidity. I present the monthly traded values in billions of Czech koruna for all government bonds used in the model estimation in the bottom

[7] The average price spread for a given date is calculated as the unweighted arithmetic average of the differences between the ask and bid price quotes of all bonds that are used in the estimation on that date.

Figure 5 The top chart displays the average price spread between the ask and bid prices of government bonds. The bottom chart displays the monthly traded values of government bonds in billions of Czech koruna. Both charts only consider government bonds used in the model estimation.

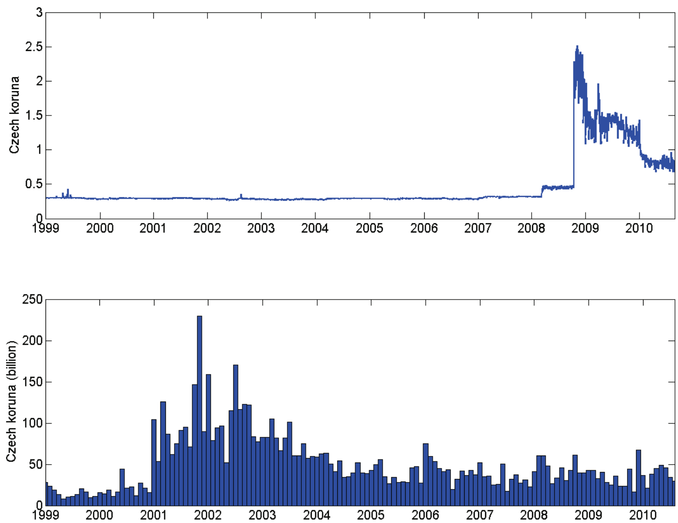


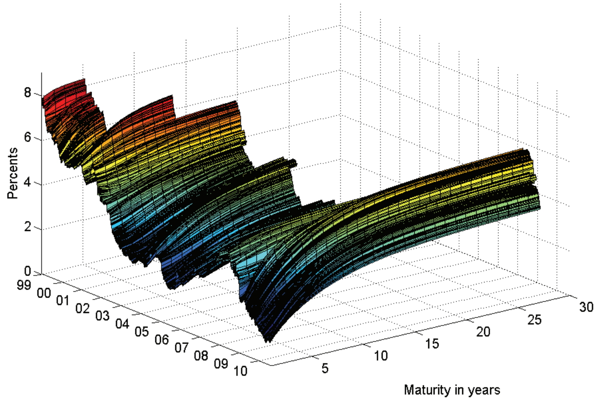
chart of *Figure 5*. The traded values are provided by the Prague Stock Exchange and cover the vast majority of Czech government bond trades. An increase in the traded values at the beginning of 2001 is apparent from this chart. The increase in traded values may explain the improvement of the error measures at the beginning of 2001 because the higher market activity might have decreased the pricing anomalies across various bonds. However, the traded values did not, surprisingly, decrease during the financial crisis compared to the last three pre-crisis years. Thus, the significance of the relation between the error measures and the traded values is not robust on different subsamples of our data set.

6.3 Individual Yield to Maturity Errors

I dig deeper into the estimation error and plot the evolution of the yield to maturity errors for each bond used in the estimation in *Figure 9* in *Appendix B*. As we estimate the yield curve day by day and do not take into account the time series behavior of the yield curve, which is strongly persistent for all maturities, the estimation errors are strongly autocorrelated. The averages of the lag 1 (one day), lag 5 (one week), and lag 20 (one month) sample autocorrelations are 0.91, 0.79, and 0.52, respectively. It is desirable that the errors oscillate around zero, i.e., that they do not exhibit any systematic behavior.[8] Some bonds show relatively long intervals when the estimation error stays constantly positive or negative (issue numbers 25, 29, 36, 46, 49, 50, 52, and 58). This is unwelcome, but because of the small number of bonds available, I have decided not to exclude these bonds from the estimation. Again, it is

[8] Systematic overpricing was the reason for excluding the 6.08%/2001 bond (issue number 27) from the estimation (see Section 4.1).

Figure 6 Estimated zero-coupon yield curves (continuously compounded). For each curve, the shortest maturity is one year and the longest maturity is given by the maximum maturity of the bond used in the estimation.



apparent from *Figure 9* that since the current financial market crisis the errors have been rising for every outstanding bond. I do not present the price error measures or individual price errors in this paper, since they do not bring any new insight into the estimation results. I can provide them upon request.

6.4 Estimated Spot Rates

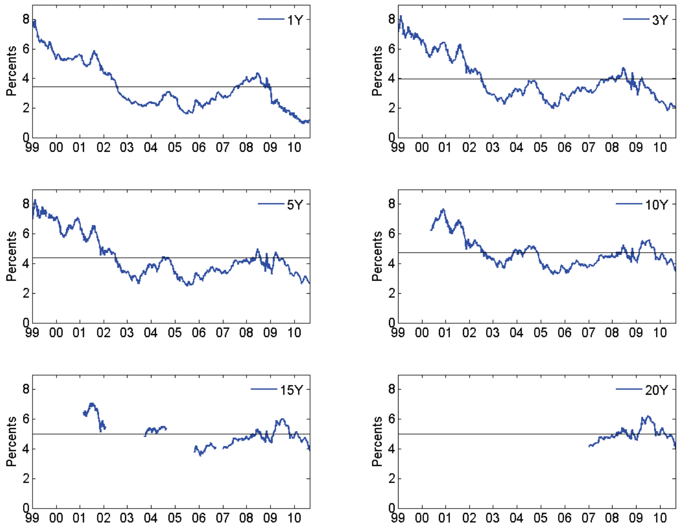
From the estimated parameters, I calculate spot rates and hence forward and par rates for each day of our data set. The estimated spot, forward, and par rates are posted in the data appendix on the web site of this journal. I provide a three-dimensional plot of the estimated spot rates in *Figure 6*.

The three-dimensional plot presents the estimated spot yield curves, day after day, from January 4, 1999 to August 24, 2010. For each yield curve, the shortest maturity is one year and the longest maturity is given by the maximum maturity of the bond used in the estimation, or more precisely, the maximum maturity of the bond is rounded to the nearest integer toward infinity. Of course, it is possible to calculate spot rates for maturities beyond the maturity range used in the estimation.

Furthermore, I slice the three-dimensional plot to capture the time series of the estimated spot rates for selected maturities in *Figure 7*. The blank space in the time series occurs when the maturity is not covered by the bonds used in the estimation. The horizontal line represents the unconditional mean of the rates.

Since 1999, the Czech spot rates exhibit a downward trend with some upward spikes and an approximate one-year upward trend (from mid-2003 to mid-2004) before bottoming out in mid-2005. From mid-2005 to mid-2008, the rates revert back to their unconditional means. This supports the mean-reverting property of interest rates – one of the most important stylized facts of interest rate behavior. It also gives some hope for rejection of the unit root in interest rate processes, which is an essential requirement for basically any interest rate time series modeling. In the first quarter of 2010, the rates start to decrease once again.

Figure 7 Estimated spot rates (continuously compounded) for selected maturities. The horizontal line represents the unconditional mean of the rates.



The slope, measured as the 5-year rate the minus 1-year rate, has been positive since the beginning of February 1999. The yield curve is inverted for the maturity range 1 to 5 years only in January 1999. At the beginning of 2009, the slope of the yield curve increases very sharply, as the 1-year rate decreases and the 5-year rate quickly rises. The slope stays at unprecedentedly high levels until the end of our data set (see *Figure 8A* in Appendix on the web site of this journal).

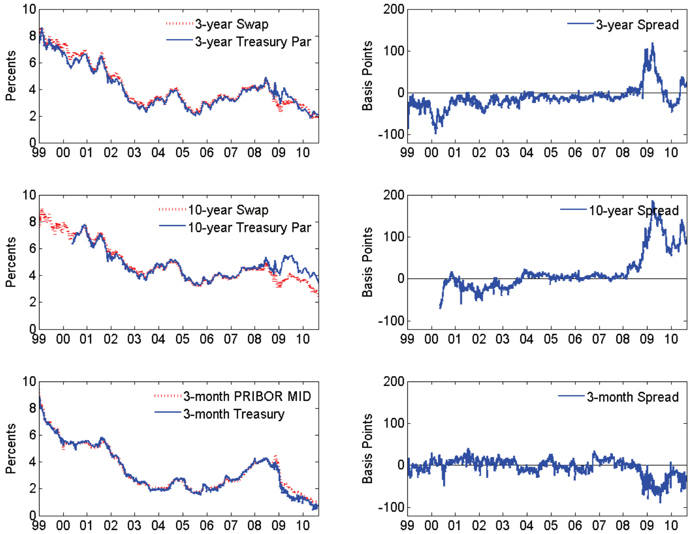
6.5 Estimated Par Rates versus Swap Rates

As mentioned in Section 2.2, interest rate swaps are an example of par rates quoted on the financial markets. The swap yield curve is often considered to be another benchmark yield curve (besides the Treasury yield curve) of the economy. The swap rates and Treasury rates can be compared with each other to measure the credit risk (the possibility of default) of the country. A swap contract entails some credit risk, but the potential losses from defaults on a swap are much smaller than the potential losses from defaults on a bond with the same notional principal. This is because the value of the swap is usually only a small fraction of the value of the bond.

I transform the estimated spot rates into discount functions according to (2) then bootstrap using (8) to get the Treasury par rates. I use end-of-day swap rate quotes from Bloomberg. The swap rates and PRIBOR rates use the ACT/360 day-count convention. Thus, I multiply the swap rate quotes by 365/360 to adjust them approximately to the 30E/360 day-count convention for bonds. I plot the Treasury par rates versus swap rates and their spreads in *Figure 8*.

The first two left-hand charts plot the 3-year and 10-year swaps together with the corresponding Treasury par rates. The bottom chart plots the 3-month Treasury rate and the 3-month PRIBOR MID rate. As the Treasury and swap rate time series

Figure 8 **The first two left-hand charts show the 3-year and 10-year swaps together with the corresponding Treasury par rates. The bottom chart shows the 3-month Treasury rate together with the corresponding 3-month PRIBOR MID rate. All rates are annually compounded. The right-hand charts provide a detailed look at the corresponding spreads.**



copy each other very closely, I present the corresponding spreads in the right-hand charts. The spreads, calculated as the Treasury rate minus the swap rate, document several issues.

First, the estimated Treasury par rates follow the dynamics of the quoted swap rates very precisely for all maturities, which confirms the quality of the estimated Treasury rates.

Second, the spreads rise substantially during the financial market crisis. The spreads had already started to rise in March 2008. For example, the average 10-year spread is 4.6 basis points for February 2008, whereas it is 23 basis points for March 2008. The 10-year spread reaches its all-time maximum of 186.7 basis points on March 23, 2009, shows a decreasing trend until the end of 2009, and rises again during the first half of 2010. The widening spread opens a debate about which yield curve should be considered as the risk-free benchmark yield curve.[9] In my view, the Czech Treasury curve is the principal risk-free benchmark curve of the Czech economy. The reason is straightforward: an interest rate swap is just an agreement to exchange payments; it is not an instrument for investing money. Money must be invested in some asset and Czech Treasury bonds naturally have the lowest credit risk among any investment instruments denominated in Czech koruna.

Third, the lows of the spread between the 3-month Treasury rate and the 3-month PRIBOR MID rate, which are present since the end of 2008, indicate an increase in the credit/liquidity premium on the Czech money market during the financial crisis.

[9] For example, this issue is important for the Czech Society of Actuaries. The risk-free benchmark yield curve is used in the liability adequacy test. For an ongoing discussion on this topic visit http://www.actuaria.cz.

7. Conclusion

I have estimated the Czech Treasury yield curve from 1999 to the present at a daily frequency. I rely on the simple and parsimonious model of Nelson and Siegel (1987), for which I suggest a lower-bound restriction on one of the model parameters. This restriction prevents abrupt changes in the parameter estimates and thus allows for the economic interpretation of the model to hold. The model fits the Czech government bond price data well without being over-parameterized and thus provides an accurate and consistent picture of the Czech Treasury yield curve evolution.

I pay close attention to estimation issues. I run a simulation exercise to verify that the suggested estimation strategy provides credible parameter estimates. I analyze the estimation error, discuss the impact of the recent financial crisis, and comment on the Czech Treasury yield curve evolution.

Despite being a fundamental economic variable, the estimated Czech Treasury yield curve is not available. I fill this gap and create time series of spot and hence forward and par interest rates, extending back into the past as much as possible. The data appendix on the web site of this journal provides estimated spot rates (continuously compounded), instantaneous forward rates (continuously compounded), and par rates (coupon equivalent) at daily frequency from January 4, 1999. The estimated parameters are also included in the data appendix, so interest rates of practically any maturity can be calculated. The Matlab implementation of the Nelson-Siegel and Svensson model is available upon request from the author.

APPENDIX

A. Czech Government Bonds Used in Yield Curve Estimation

Table 3 lists all bonds used in the estimation of the Czech Treasury yield curve from 1999 to the present. The issue number is a unique identifier which is incremented by one for every new Czech government bond issue. The total amount issued is reported in billions of Czech koruna. Note that the Ministry of Finance has been buying government bonds in the primary auctions on its own account in recent years. Thus, the total amount issued does not necessarily exactly match the total amount sold to primary dealers in the primary auctions. Note also that the total amount issued will rise further for the 3.40%/2015 bond (issue number 60), the 2.80%/2013 bond (issue number 59), the 5.70%/2024 bond (issue number 58), and the 5.00%/2019 bond (issue number 56), as these bond issues are still being re-opened at the time of writing this paper.

B. Yield to Maturity Errors for Individual Bonds

Figure 9 displays the yield to maturity errors in basis points, i.e., the residuals between the observed and the fitted (model-implied) yields to maturity. The bond can be identified by the issue number shown in the legend.

Table 3

ISSIN	Issue Date	Maturity Date	Coupon Rate (%)	Issue Number	Total Amount Issued (CZK bn)
CZ0001002737	01/03/2010	01/09/2015	3.4	60	30
CZ0001002729	01/02/2010	16/09/2013	2.8	59	33
CZ0001002547	25/05/2009	25/05/2024	5.7	58	61
CZ0001002471	23/03/2009	11/04/2019	5.0	56	87
CZ0001002158	28/01/2008	11/04/2011	4.10	54	56
CZ0001001945	18/06/2007	12/09/2022	4.70	52	55
CZ0001001903	30/04/2007	11/04/2017	4.00	51	69
CZ0001001887	16/04/2007	18/10/2012	3.55	50	70
CZ0001001796	04/12/2006	04/12/2036	4.20	49	20
CZ0001001754	27/11/2006	27/11/2009	3.25	48	50
CZ0001001309	26/09/2005	26/09/2008	2.30	47	49
CZ0001001317	12/09/2005	12/09/2020	3.75	46	69
CZ0001001242	18/07/2005	18/10/2010	2.55	45	57
CZ0001001143	11/04/2005	11/04/2015	3.80	44	62
CZ0001000863	02/08/2004	02/08/2007	3.95	43	23
CZ0001000855	22/03/2004	22/03/2009	3.80	42	49
CZ0001000822	18/08/2003	18/08/2018	4.60	41	47
CZ0001000814	16/06/2003	16/06/2013	3.70	40	65
CZ0001000798	17/03/2003	17/03/2008	2.90	39	41
CZ0001000780	20/01/2003	20/01/2006	3.00	38	30
CZ0001000772	26/10/2001	26/10/2006	5.70	37	28
CZ0001000764	05/10/2001	05/10/2011	6.55	36	50
CZ0001000756	14/09/2001	14/09/2004	6.05	35	29
CZ0001000749	26/01/2001	26/01/2016	6.95	34	35
CZ0001000731	14/04/2000	14/04/2010	6.40	33	21
CZ0001000723	17/03/2000	17/03/2007	6.30	32	20
CZ0001000707	18/02/2000	18/02/2005	6.75	31	22
CZ0001000715	05/02/2000	05/02/2004	7.95	26	5
CZ0001000681	21/01/2000	21/01/2003	6.90	30	18
CZ0001000640	05/11/1999	05/11/2001	6.50	29	7
CZ0001000632	06/08/1999	06/08/2004	7.30	28	5
CZ0001000582	06/11/1998	06/11/2000	10.85	25	5
CZ0001000574	07/08/1998	07/08/2003	10.90	24	5
CZ0001000566	15/05/1998	15/05/2000	14.75	23	5
CZ0001000558	06/02/1998	06/02/2003	14.85	22	5

Figure 9a

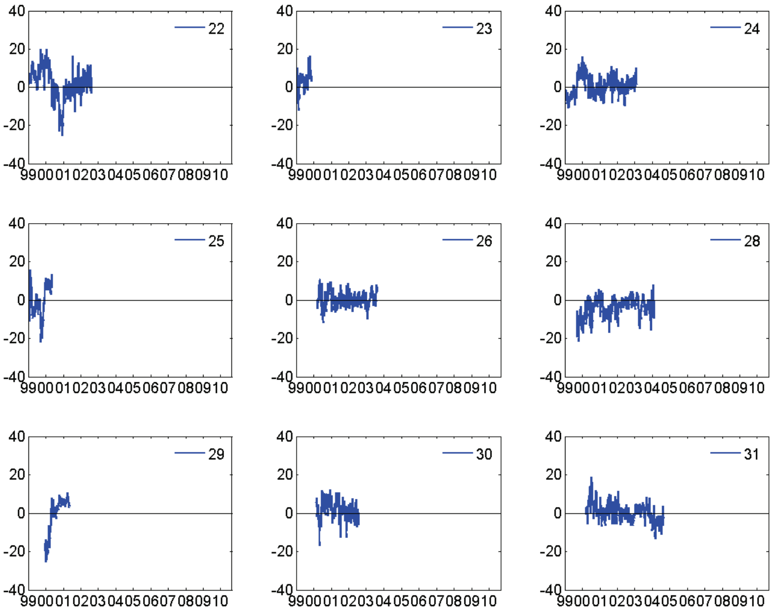


Figure 9b

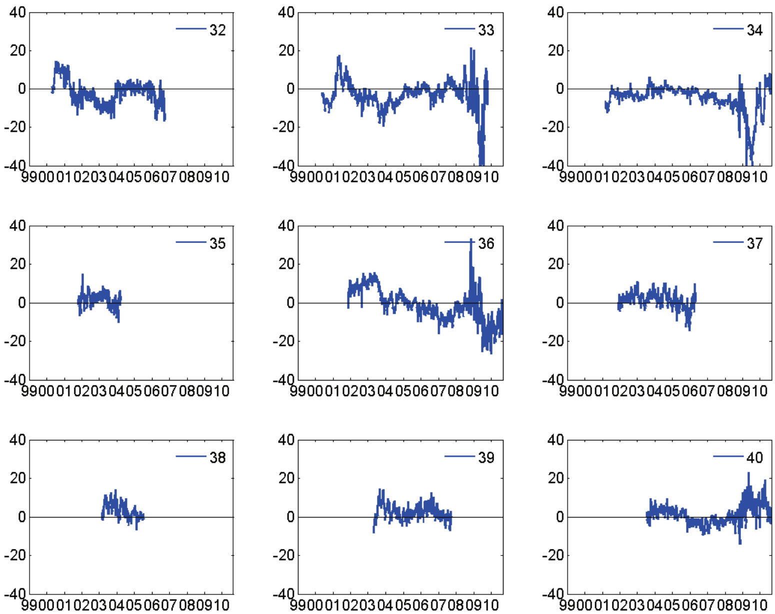


Figure 9c

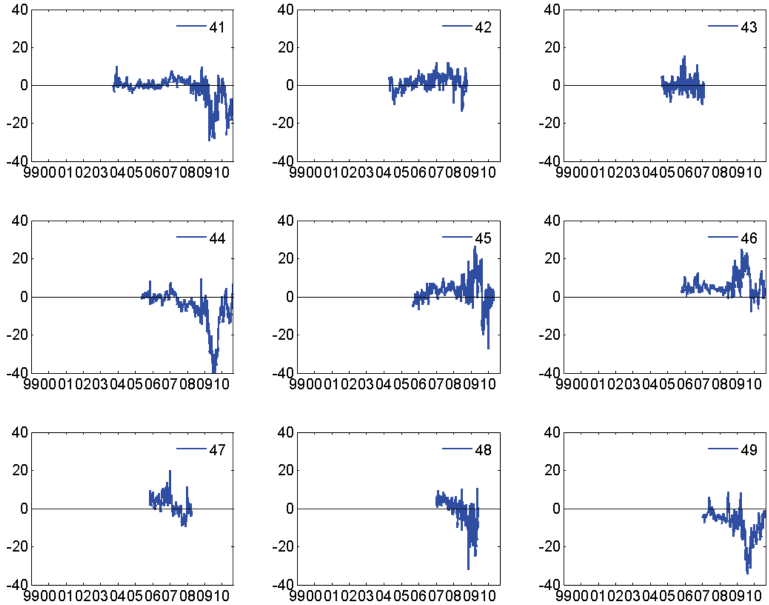
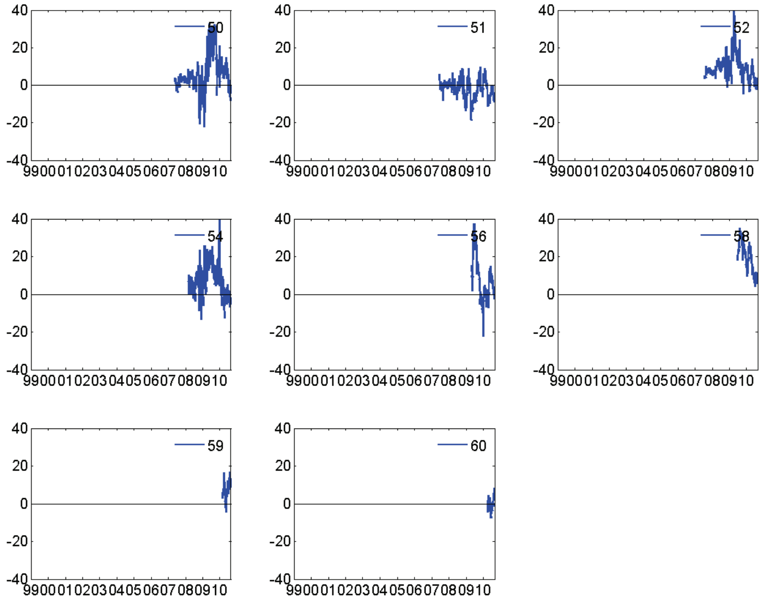


Figure 9d



REFERENCES

BIS (2005): Zero-Coupon Yield Curves: Technical Documentation. Basle, Bank for International Settle-ments, *BIS Paper*, no. 25.

Cairns AJG (2004): *Interest Rate Models: An Introduction.* Princteon University Press, New Jersey.

Cairns AJG, Pritchard DJ (2001): Stability of descriptive models for the term structure of interest rates with application to German market data. *British Actuarial Journal*, 17:467–507.

Christensen JHE, Diebold FX, Rudebusch GD (2010): The Affine Arbitrage-Free Class of Nelson--Siegel Term Structure Models. *Journal of Econometrics*, Forthcoming.

Cipra T (2000): *Matematika cenných papírů.* HZ Praha, Praha.

Coleman TF, Li Y (1996): An Interior, Trust Region Approach for Nonlinear Minimization Subject to Bounds. *SIAM Journal on Optimization*, 6:418–445.

Diebold FX, Li C (2006): Forecasting the term structure of government bond yields. *Journal of Econometrics*, 130:337–364.

Diebold FX, Rudebusch GD, Arruoba SB (2006): The Macroeconomy and the Yield Curve: A Dynamic Latent Factor Approach. *Journal of Econometric*s, 131:309–338.

Gimeno R, Nave JM (2009): A genetic algorithm estimation of the term structure of interest rates. *Computational Statistics and Data Analysis*, 53:2236–2250.

Gurkaynak RS, Sack B, Wright JH (2007): The U.S. Treasury yield curve: 1961 to the present. *Journal of Monetary Economics*, 54:2291–2304.

Kladívko K, Cícha M, Zimmermann P (2007): *Yield curve modeling using principal component analysis and nonlinear stochastic differential equations.* Paper presented at the 2007 Mathematical Methods in Economics and Industry Conference, Herľany, June 4, 2007. Available at: http://umv.science.upjs.sk/mmei07/documents/Abstracts/KladivkoAbstract.pdf.

Málek J, Radová J, Štěrba F (2006): Konstrukce výnosové křivky pomocí vladních dluhopisů v České republice. *Politická ekonomie*, 6:792–808.

Nelson CR, Siegel AF (1987): Parsimonious Modeling of Yield Curves. *Journal of Business*, 60:473–489.

Slavík M (2001): Odhad časové struktury úrokových sazeb z cen domacích dluhopisů. *Finance a úvěr*, 51:591–606.

Svensson LEO (1995): Estimating Forward Interest Rates with Extended Nelson & Siegel Method. *Penning & Valutapolitik – Sveriges Riksbank Quarterly Review*, 3:13–26.

Waggoner DF (1997): Spline Methods for Extracting Interest Rate Curves from Coupon Bond Prices. *Federal Reserve Bank of Atlanta Working Paper*, no. 97-10.