POLYNOMIAL ASYMPTOTES AND ASYMPTOTIC SERIES

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Abstract: *The article deals with a generalization of the concept of the asymptote (with a slope) of a graph function, which is a linear function to an analogous concept - the asymptote of a graph function, which can be not only a linear function but also a higher degree polynomial function. It also generalizes this concept by asymptotic series. The contribution also presents the possibilities of using the Maxima - open source system to determine the polynomial asymptote and the asymptotic series of a given function.*

Keywords: asymptotes of graph functions, polynomial (curvilinear) asymptotes of graph functions, asymptotic series, Laurent series, Maxima - open source system.

INTRODUCTION

In the basic course of mathematics at university, the asymptote of the function graph is defined as a line such that the distance between the curve and the line approaches zero as one or both of the *x* or *y* coordinates tends to infinity [1], [3], [5], [7]. In some contexts, such as algebraic geometry, an asymptote is defined as a line which is tangent to a curve at infinity.
More generally, one curve is a polynomial (curvilinear) asymptote of another (as opposed to a linear asymptote) if the distance between the two curves tends to zero as they tend to infinity.
An asymptotic series is a series expansion of a function in a variable *x* which may converge or diverge, but whose partial sums can be made an arbitrarily good approximation to a given function for large enough *x*.

1 ASYMPTOTES OF A GRAPH FUNCTION

1.1 Linear asymptotes

Definition 1.1 We say that line $x = a$ is a *vertical asymptote* of the graph of the function $y = f(x)$ (or simply a *vertical asymptote* of the function) if at least one of the following statements is true:

$$\lim_{x\to a^+} f(x) = \infty,\ \lim_{x\to a^-} f(x) = \infty,\ \lim_{x\to a^+} f(x) = -\infty,\ \lim_{x\to a^-} f(x) = -\infty. \qquad (1)$$

Definition 1.2 [1] If there are limits

$$\lim_{x\to+\infty} \frac{f(x)}{x} = k_1, \quad k_1 \in \mathbb{R},$$

and

$$\lim_{x\to+\infty}\big(f(x)-k_1x\big)=q_1,\quad q_1\in\mathbb{R},$$

then the straight line $y=k_1x+q_1$ will be an *asymptote* (a *right inclined asymptote or, when* $k_1=0$, a *right horizontal asymptote*) of the function $y=f(x)$.

If there are limits

$$\lim_{x\to-\infty}\frac{f(x)}{x}=k_2,\quad k_2\in\mathbb{R},$$

and

$$\lim_{x\to-\infty}\big(f(x)-k_2x\big)=q_2,\quad q_2\in\mathbb{R},$$

then the straight line $y=k_2x+q_2$ is an *asymptote* (a *left inclined asymptote or, when* $k_2=0$, a *left horizontal asymptote*) of the function $y=f(x)$.

The graph of the function $y=f(x)$ cannot have more than one right (inclined or horizontal) and more than one left (inclined or horizontal) asymptote.

Example 1.1 In the figure 1 is a graph of the function $y=\dfrac{x^2-x+4}{2x+2}$ with their asymptotes $x=-1$ (vertical) and $y=\dfrac{1}{2}x-1.$

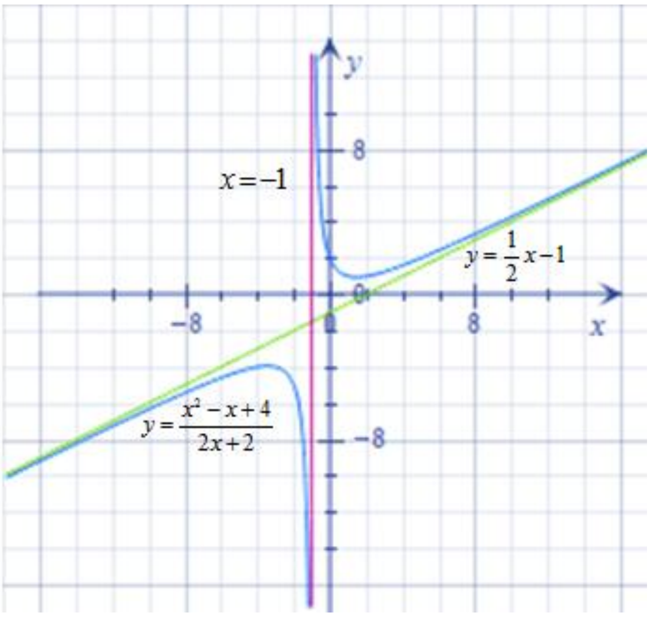

Fig. 1. Asymptotes of the graph of the function (Example 1.1).
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1.2 Polynomial (curvilinear) asymptotes

Definition 1.3 We say that polynomial function

$$P_n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0,\ a_n\neq 0, \tag{2}$$

is *a polynomial* (*curvilinear*) *asymptote* of the graph function $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x\to\infty}\left(f(x)-P_n(x)\right)=0 \text{ or } \lim_{x\to-\infty}\left(f(x)-P_n(x)\right)=0. \tag{3}$$

In the first case, we say about the *polynomial asymptote of the graph of the function* $y = f(x)$ *for* $x \to +\infty$, in the second case the *polynomial asymptote of the graph of the function* $y = f(x)$ *for* $x \to -\infty$.

Next, we will only deal with the polynomial asymptotes of the *graph of the function* for $x \to +\infty$.

Theorem 1.2 The polynomial function (2) is a polynomial asymptote of the graph of the function $y = f(x)$ if and only if there are proper limits

$$\begin{aligned}
&\lim_{x\to\infty}\frac{f(x)}{x^n}=a_n\\
&\lim_{x\to\infty}\left(\frac{f(x)}{x^{n-1}}-a_n x\right)=a_{n-1}\\
&\vdots\\
&\lim_{x\to\infty}\left(\frac{f(x)}{x^k}-a_n x^{n-k}-a_{n-1}x^{n-k-1}-\cdots-a_{k+1}x\right)=a_k\\
&\vdots\\
&\lim_{x\to\infty}\left(f(x)-a_n x^n-a_{n-1}x^{n-1}-\cdots-a_1 x\right)=a_0
\end{aligned} \tag{4}$$

Proof. Proof will be done only for $x \to +\infty$.

1. Let polynomial function (2) be a polynomial asymptote of the graph of the function f. By edit the first of the relationship (3) we have

$$\begin{aligned}
\lim_{x\to\infty}\left(f(x)-P_n(x)\right)&=\lim_{x\to\infty}\left(f(x)-\left(a_n x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\right)\right)=\\
&=\lim_{x\to\infty}x^n\left(\frac{f(x)}{x^n}-a_n-\frac{a_{n-1}}{x}-\cdots-\frac{a_1}{x^{n-1}}-\frac{a_0}{x^n}\right)=0.
\end{aligned}$$

Because it is true that $\lim\limits_{x\to\infty}x^n=\infty$, it must be necessary that

$$\lim_{x\to\infty}\left(\frac{f(x)}{x^n}-a_n-\frac{a_{n-1}}{x}-\cdots-\frac{a_1}{x^{n-1}}-\frac{a_0}{x^n}\right)=0,$$

otherwise, the first limit in relation (3) would not be proper. So that

$$\begin{aligned}
\lim_{x\to\infty}\left(\frac{f(x)}{x^n}-a_n-\frac{a_{n-1}}{x}-\cdots-\frac{a_1}{x^{n-1}}-\frac{a_0}{x^n}\right)&=\lim_{x\to\infty}\frac{f(x)}{x^n}-\lim_{x\to\infty}\left(a_n-\frac{a_{n-1}}{x}-\cdots-\frac{a_1}{x^{n-1}}-\frac{a_0}{x^n}\right)=\\
&=\lim_{x\to\infty}\frac{f(x)}{x^n}-a_n=0.
\end{aligned}$$

Then

$$\lim_{x\to\infty}\frac{f(x)}{x^n}=a_n,$$

which is the first of the relationships (4). Similarly, we proceed further.

Let us compute

$$\lim_{x\to\infty}\left(f(x)-P_n(x)\right)=\lim_{x\to\infty}\left(f(x)-a_nx^n-a_{n-1}x^{n-1}-\cdots-a_1x-a_0\right)=$$
$$=\lim_{x\to\infty}x^{n-1}\left(\frac{f(x)}{x^{n-1}}-a_nx-a_{n-1}-\frac{a_{n-2}}{x}-\cdots-\frac{a_1}{x^{n-2}}-\frac{a_0}{x^{n-1}}\right)=0.$$

Then

$$\lim_{x\to\infty}\left(\frac{f(x)}{x^{n-1}}-a_nx-a_{n-1}-\frac{a_{n-2}}{x}-\cdots-\frac{a_1}{x^{n-2}}-\frac{a_0}{x^{n-1}}\right)=\lim_{x\to\infty}\left(\frac{f(x)}{x^{n-1}}-a_nx\right)-a_{n-1}-$$
$$-\lim_{x\to\infty}\left(\frac{a_{n-2}}{x}+\cdots+\frac{a_1}{x^{n-2}}+\frac{a_0}{x^{n-1}}\right)=\lim_{x\to\infty}\left(\frac{f(x)}{x^{n-1}}-a_nx\right)-a_{n-1}-0=0,$$

and from there

$$\lim_{x\to\infty}\left(\frac{f(x)}{x^{n-1}}-a_nx\right)=a_{n-1}$$

which is the second of the relationships (4).
Similarly, all relationships in (4) can be derived. Note that the last relation in (4), which expresses the absolute coefficient, is calculated directly from the condition

$$\lim_{x\to\infty}\left(f(x)-\left(a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\right)\right)=0$$

So that

$$\lim_{x\to\infty}\left(f(x)-a_nx^n-a_{n-1}x^{n-1}-\cdots-a_1x\right)=a_0\,.$$

2. We will prove opposite implication. Assume there are proper limits (4) to compute $a_n, a_{n-1},\ldots,a_1,a_0$. Then from the last relationship (4)

$$\lim_{x\to\infty}\left(f(x)-a_nx^n-a_{n-1}x^{n-1}-\cdots-a_1x\right)=a_0\,,$$

we get

$$\lim_{x\to\infty}\left(f(x)-a_nx^n-a_{n-1}x^{n-1}-\cdots-a_1x-a_0\right)=0,$$

It follows from definition 1.3, that the polynomial function

$$P_n(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

with such calculated coefficients is the polynomial asymptote of the function graph f .

Consequence 1.1 There is always a polynomial asymptote for every rational function.

Proof. Let us consider a rational function

$$R(x)=\frac{P(x)}{Q(x)}\,, \tag{5}$$

where $P(x)$ and $Q(x)$ are polynomials.
If a rational function is a proper rational function, that the degree of the numerator $P(x)$ is lower than that of the nominator $Q(x),$ then polynomial asymptote is obviously the horizontal asymptote

$$y=0.$$

On the other hand, there exist single-valued polynomials $U(x)$ and $Z(x)$, where the degree of the polynomial $Z(x)$ is lower than the degree of the polynomial $Q(x),$ that [5]

$$R(x)=\frac{P(x)}{Q(x)}=U(x)+\frac{Z(x)}{Q(x)}.$$

Then polynomial function $U(x)$ is a polynomial asymptote of the graph of the function $R(x)$ because

$$\lim_{x\to\infty}\left(R(x)-U(x)\right)=\lim_{x\to\infty}\frac{R(x)}{Q(x)}=0.$$

Example 1.2 The function $y=x^2$ is a polynomial asymptote of the graph of the function $R(x)=\dfrac{x^4+x^2+5}{x^2+1}$ (see figure 2), because $\dfrac{x^4+x^2+5}{x^2+1}=x^2+\dfrac{5}{x^2+1}$ and

$$\lim_{x\to\pm\infty}\left(\frac{x^4+x^2+5}{x^2+1}-x^2\right)=\lim_{x\to\pm\infty}\frac{5}{x^2+1}=0.$$

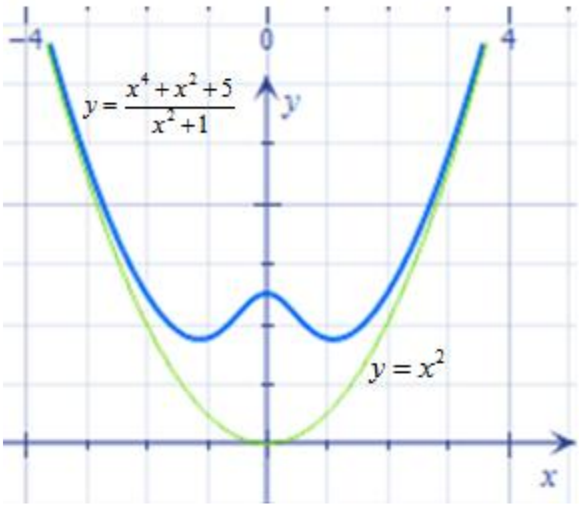

Fig. 2. The polynomial asymptote of the graph of the function (Example 1.2).
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Example 1.3 Find the polynomial asymptote of the graph of the function

$$f(x)=\sqrt{4x^4+4x^3+9x^2+1},$$

for $x\to+\infty$.

Solution. We seek quadratic asymptotes

$$y=a_2x^2+a_1x+a_0$$

of the graph of the function *f*, for $x\to+\infty$. According theorem 1.2 we obtain

$$a_2=\lim_{x\to\infty}\frac{f(x)}{x^2}=\lim_{x\to\infty}\frac{\sqrt{4x^4+4x^3+9x^2+1}}{x^2}=2.$$

Remark that

$$\lim_{x\to\infty}\frac{f(x)}{x^n}=0 \text{ for } n>2.$$

Further

$$a_1=\lim_{x\to\infty}\left(\frac{f(x)}{x}-2x\right)=\lim_{x\to\infty}\left(\frac{\sqrt{4x^4+4x^3+9x^2+1}}{x}-2x\right)=1,$$

$$a_0 = \lim_{x\to\infty}\left(f(x) - 2x^2 - x\right) = \lim_{x\to\infty}\left(\sqrt{4x^4+4x^3+9x^2+1} - 2x^2 - x\right) = 2.$$

The sought quadratic asymptote is

$$y = 2x^2 + x + 2. \qquad \square$$

2 ASYMPTOTIC SERIES

2.1 Asymptotic series of a function

Asymptotic series of a function f is usually defined [2], [6] as a series

$$\sum_{k=0}^{\infty} \frac{a_k}{x^k}, \tag{6}$$

(at convention $x^0 = 1$), which satisfied the condition

$$\lim_{x\to\infty} x^n R_n(x) = \lim_{x\to\infty} x^n \left(f(x) - S_n(x)\right) = 0 \text{ for fixed } n$$

where

$$S_n(x) = \sum_{k=0}^{n} \frac{a_k}{x^k}.$$

Example 2.1 Find an asymptotic series of the function $f(x) = \dfrac{1}{x-1}$ and its domain of convergence.

Solution.

$$\frac{1}{x-1} = \frac{\frac{1}{x}}{1-\frac{1}{x}} = \frac{1}{x}\left(1 + \frac{1}{x} + \frac{1}{x^2} + \cdots\right) = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots, \ x \neq 0.$$

The domain of convergence:

$$\left|\frac{1}{x}\right| < 1 \iff |x| > 1 \iff x \in (-\infty, -1) \cup (1, \infty).$$

In $x=-1$ and $x=1$ the series is divergent. $\square$

However, we are under asymptotic series of the function f for some $|x| > x_0,\ x_0 > 0$ (if series converges) we will further understand Laurent series with the finite regular part, of the form

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 + \frac{a_{-1}}{x} + \frac{a_{-2}}{x^2} + \frac{a_{-3}}{x^3} + \cdots, \ a_n \neq 0. \tag{7}$$

The coefficients of this series (if exist) are determined in a manner analogous to polynomial asymptotes (see also [5], p. 535, [8], [9]). The procedure will be interpreted to identify several members of the asymptotic series of the function from Example 1.3.

Example 2.2 Find several members of the asymptotic series of the function

$$f(x) = \sqrt{4x^4+4x^3+9x^2+1}$$

Solution. We find an asymptotic series of the function in the form

$$f(x) = a_2x^2 + a_1x + a_0 + \frac{a_{-1}}{x} + \frac{a_{-2}}{x^2} + \frac{a_{-3}}{x^3} + \cdots \ . \tag{8}$$

We determine the coefficients $a_2,\ a_1,\ a_0$ using the same procedure as we set them in Example 1.3. The coefficient a_{-1} we determine by substitute into the relation (8) for already calculated values $a_2,\ a_1,\ a_0$ and multiplying equation by *x*, partially editing, and limiting both sides of the equation for $x \to +\infty$.

We obtain

$$\lim_{x\to\infty} x\left(f(x) - 2x^2 - x - 2\right) = \lim_{x\to\infty}\left(a_{-1} + \frac{a_{-2}}{x} + \frac{a_{-3}}{x^2} + \frac{a_{-4}}{x^3}\cdots\right) = a_{-1}.$$

Because

$$\lim_{x\to\infty} x\left(f(x) - 2x^2 - x - 2\right) = -1, \text{ and so } a_{-1} = -1.$$

We calculate a_{-2}. The Equation (8) we multiply by x^2, simplify it and limit both sides.

We get

$$\lim_{x\to\infty} x^2\left(f(x) - 2x^2 - x - 2 + \frac{1}{x}\right) = \lim_{x\to\infty}\left(a_{-2} + \frac{a_{-3}}{x} + \frac{a_{-4}}{x^2}\cdots\right) = a_{-2}.$$

Because

$$\lim_{x\to\infty} x^2\left(f(x) - 2x^2 - x - 2 + \frac{1}{x}\right) = -\frac{1}{4}, \text{ and so } a_{-2} = -\frac{1}{4}.$$

Similarly, we would calculate other coefficients of the series. We would have that

$$\sqrt{4x^4 + 4x^3 + 9x^2 + 1} = 2x^2 + x + 2 - \frac{1}{x} - \frac{1}{4x^2} + \frac{9}{8x^3} - \frac{9}{16x^4} - \frac{31}{32x^5} + \cdots =$$
$$= 2x^2 + x + 2 - \frac{1}{x} - \frac{1}{4x^2} + O\left(\left(\frac{1}{x}\right)^3\right).$$

So we can see that regular part of the Laurent series is polynomial asymptotes of the graph of the function.

2.2 Taylor expansion and asymptotic series

Without the exact proof we will outline the possibilities of using Taylor's series [4], [5], [8] to determine the asymptotic series (if exists) of a given function (for $x \to +\infty$), if there exists. We will interpret the procedure on the function from Example 1.3. Substitute in the expression

$$\sqrt{4x^4 + 4x^3 + 9x^2 + 1}$$

$x = \frac{1}{t}$ and simplification of obtained expression

$$\sqrt{4\frac{1}{t^4}+4\frac{1}{t^3}+9\frac{1}{t^2}+1}=\frac{\sqrt{t^4+9t^2+4t+4}}{t^2}.$$

It is clear that the function $y=\frac{\sqrt{t^4+9t^2+4t+4}}{t^2}$ cannot be expanded in some neighborhood $|x|<R$ of the point 0 in the Taylor series because function is not defined in $t=0$. However, we can expand the function $y=\sqrt{t^4+9t^2+4t+4}$ in the Taylor series in some neighborhood $|x|<R$ of the point 0. We get

$$\sqrt{t^4+9t^2+4t+4}=2+t+2t^2-t^3-\frac{t^4}{4}+\frac{9t^5}{8}-\frac{9t^6}{16}-\frac{31t^7}{32}+\frac{51t^8}{32}+\cdots \tag{9}$$

After multiplying (9) by $\frac{1}{t^2}$ we get for $t\neq 0$

$$\frac{\sqrt{t^4+9t^2+4t+4}}{t^2}=\frac{2}{t^2}+\frac{1}{t}+2-t-\frac{t^2}{4}+\frac{9t^3}{8}-\frac{9t^4}{16}-\frac{31t^5}{32}+\frac{51t^6}{32}+\cdots$$

By returning the substitution $t=\frac{1}{x}$ and editing expression, we get asymptotic series of a function

$$\sqrt{4x^4+4x^3+9x^2+1}=2x^2+x+2-\frac{1}{x}-\frac{1}{4x^2}+\frac{9}{8x^3}-\frac{9}{16x^4}-\frac{31}{32x^5}+\frac{51}{32x^6}+\cdots$$

Again we can see that first part of the series is a polynomial asymptote of the graph of the function $f(x)=\sqrt{4x^4+4x^3+9x^2+1}$.

If we wanted to precise of the approximation of the function f for calculate sufficiently large *x*, we would add several more members of the asymptotic series. □

3 AVAILABLE SOFWARES FOR DETERMINING POLYNOMIAL ASYMPTOTES AND ASYMPTOTIC SERIES

3.1 Possibilities of using Maxima

The open source system Maxima provides more opportunity for determining of the asymptotic series of a function and to determine a polynomial asymptote from it. These are commands with examples

taylor (expr, x, a, n) expands the expression expr in a truncated Taylor or Laurent series in the variable x around the point a, containing terms through (x - a)^n. If $a=\infty$, then we get Laurent series.

(%i1) taylor (sqrt(1+9*x^2+4*x^3+4*x^4), x, inf, 5);

(%o1)/T/ $2x^2+x+2-\frac{1}{x}-\frac{1}{4x^2}+\frac{9}{8x^3}-\frac{9}{16x^4}-\frac{31}{32x^5}+\ldots$

returns an expansion of expr in negative powers of x - a. The highest order term is (x - a)^-n.

taylor (expr, [x, a, n, 'asymp])

(%i2) taylor (sqrt(1+9*x^2+4*x^3+4*x^4), [x, 0, 5, 'asymp]);

$$(\%o2)/T/\ 2x^2+x+2-\frac{1}{x}-\frac{1}{4x^2}+\frac{9}{8x^3}-\frac{9}{16x^4}-\frac{31}{32x^5}+\ldots$$

powerseries (expr, x, a) returns the general form of the power series expansion for *expr* in the variable *x* about the point *a* (which may be inf for infinity).

(%i3) powerseries ((1+2*x-3*x^2+4*x^3)/(1+x), x, inf);

$$(\%o3)\ 8\left(\sum_{i1=0}^{\infty}\frac{(-1)^{i1}}{x^{i1}}\right)+4x^2-7x+1$$

3.2 Possibilities of using WolframAlpha

On the webpage

WolframAlpha

https://www.wolframalpha.com/examples/math/calculus/coordinate-geometry/asymptotes/

we can calculate polynomial asymptotes of functions and on the https://www.wolframalpha.com/examples/math/calculus/series-expansions/

is possible calculate asymptotic series of functions.

CONCLUSION

The article clearly presents the methods of calculating polynomial (curvilinear) asymptotes of functions as well as asymptotic series of functions by the gradual calculation of their coefficients. It shows the relationship between asymptotic series and polynomial asymptotes. It sketches the possibilities of using Taylor's expansion to determine asymptotic series and thus polynomial asymptotes of a function. And finally shows the possibilities of using Maxima – open source system and WolframAlpha for their determination.

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