Some Remarks On Faster Convergent Infinite Series II

DUŠAN HOLÝ [a]

[a] *Faculty of Industrial Technologies in Púchov*
Trenčín University of Alexander Dubček in Trenčín
I. Krasku 491/30, 02001 Púchov, Slovakia

LADISLAV MATEJÍČKA [b,*]

[b] *Faculty of Industrial Technologies in Púchov*
Trenčín University of Alexander Dubček in Trenčín
I. Krasku 491/30, 02001 Púchov, Slovakia

ĽUDOVÍT PINDA [c]

[c] *Faculty of Informatics, University of Economics*
Dolnozemská 1, 852 35, Bratislava, Slovakia

Abstract

Next necessary and sufficient conditions for the existence of faster convergent series with different types of their terms are found. A faster convergence of certain Kummer's series is proved in this paper.

Key words: faster convergent series, terms of convergent series
1991 MSC: 65B10, 40A10

[1]

* Corresponding author. Address: Faculty of Industrial Technologies in Púchov
Trenčín University of Alexander Dubček in Trenčín
I. Krasku 491/30, 02001 Púchov, Slovakia

Email addresses: holy@fpt.tnuni.sk, matejicka@tnuni.sk (LADISLAV MATEJÍČKA), pinda@dec.euba.sk (ĽUDOVÍT PINDA).

[1] This work was supported by VEGA no.1/0094/08.

1 Introduction and preliminaries

The goal of this paper is another generalization of the affirmation that Kummer's series are faster convergent, and especially, the elimination of the condition $\lim\limits_{n\to\infty}\frac{\triangle s_n(a)}{\triangle s_n(c)}\neq 0$. Accordingly to these facts, in this paper we study faster convergence of infinite series $\sum\limits_{n=1}^{\infty}a_n$ than $\sum\limits_{n=1}^{\infty}b_n$ without the condition $\lim\limits_{n\to\infty}\frac{\triangle s_n(a)}{\triangle s_n(b)}=0$. First we show that there exist convergent infinite series $\sum\limits_{n=1}^{\infty}a_n$, $\sum\limits_{n=1}^{\infty}b_n$ such that $\sum\limits_{n=1}^{\infty}a_n$ is faster convergent than $\sum\limits_{n=1}^{\infty}b_n$ and either $\lim\limits_{n\to\infty}\frac{\triangle s_n(a)}{\triangle s_n(b)}=c\neq 0$ or $\lim\limits_{n\to\infty}\frac{\triangle s_n(a)}{\triangle s_n(b)}=\infty$ or $\lim\limits_{n\to\infty}\frac{\triangle s_n(a)}{\triangle s_n(b)}$ does not exist. It is shown that for a certain set of faster convergent series $\sum\limits_{n=1}^{\infty}a_n$ than a given series $\sum\limits_{n=1}^{\infty}b_n$, the series satisfying the condition $\lim\limits_{n\to\infty}\frac{\triangle s_n(a)}{\triangle s_n(b)}=0$ are the best of the given set.
Strictly speaking: if $\sum\limits_{n=1}^{\infty}a_n$ is faster convergent than $\sum\limits_{n=1}^{\infty}b_n$, $\sum\limits_{n=1}^{\infty}c_n$ is faster convergent than $\sum\limits_{n=1}^{\infty}b_n$, $\lim\limits_{n\to\infty}\frac{\triangle s_n(a)}{\triangle s_n(b)}=0$ and either $\lim\limits_{n\to\infty}\frac{\triangle s_n(c)}{\triangle s_n(b)}\neq 0$ or $\lim\limits_{n\to\infty}\frac{\triangle s_n(c)}{\triangle s_n(b)}$ does not exist, then $\sum\limits_{n=1}^{\infty}a_n$ is faster convergent than $\sum\limits_{n=1}^{\infty}c_n$ (Lemma 6). In Lemmas 8-10 we found equivalent conditions for the existence of faster convergent infinite series $\sum\limits_{n=1}^{\infty}a_n$ for a given series $\sum\limits_{n=1}^{\infty}b_n$ such that either $\lim\limits_{n\to\infty}\frac{\triangle s_n(a)}{\triangle s_n(b)}=c\neq 0$ or $\lim\limits_{n\to\infty}\frac{\triangle s_n(a)}{\triangle s_n(b)}$ does not exist or $\lim\limits_{n\to\infty}\frac{\triangle s_n(a)}{\triangle s_n(b)}=\infty$. The consequences of these Lemmas are presented in Propositions 11, 13-16. The most important consequence is Lemma 17, which says that the Kummer's series $\sum\limits_{n=1}^{\infty}b_n$ are faster convergent than $\sum\limits_{n=1}^{\infty}a_n$ for a certain set of convergent infinite series $\sum\limits_{n=1}^{\infty}a_n$, $\sum\limits_{n=1}^{\infty}c_n$, despite of the fact that the $\lim\limits_{n\to\infty}\frac{\triangle s_n(c)}{\triangle s_n(a)}$ need not exist.

We denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. In what follows, if we say that $\lim\limits_{n\to\infty}a_n$ exists, we admit also the cases $\lim\limits_{n\to\infty}a_n=+\infty\ (-\infty)$ and we will suppose that terms of all infinite series are real nonzero numbers.

Definition 1 *[2] Let $\sum\limits_{n=1}^{\infty}a_n$, $\sum\limits_{n=1}^{\infty}b_n$ be convergent series such that $s(b)-s_{n-1}(b)\neq 0$, $n\in\mathbb{N}$. The series $\sum\limits_{n=1}^{\infty}a_n$ is called faster convergent series than $\sum\limits_{n=1}^{\infty}b_n$ if $\lim\limits_{n\to\infty}\frac{s(a)-s_{n-1}(a)}{s(b)-s_{n-1}(b)}=0$.*

We will write "fcst" instead of "faster convergent series than".

Lemma 2 *[10] Let* $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ *be convergent series with positive terms. If* $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = 0$ *then* $\sum_{n=1}^{\infty} a_n$ *is fcst* $\sum_{n=1}^{\infty} b_n$.

Lemma 3 *[4] Let* $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ *be convergent real series. Let* $s(b)-s_{n-1}(b) \neq 0$ *for all* $n \in \mathbb{N}$. *Let* $l_i(a) = \liminf_{n\to\infty} \left|\frac{s(a)-s_{n-1}(a)}{s_n(a)-s_{n-1}(a)}\right|$, $l_s(a) = \limsup_{n\to\infty} \left|\frac{s(a)-s_{n-1}(a)}{s_n(a)-s_{n-1}(a)}\right|$, $l_i(b) = \liminf_{n\to\infty} \left|\frac{s(b)-s_{n-1}(b)}{s_n(b)-s_{n-1}(b)}\right|$, $l_s(b) = \limsup_{n\to\infty} \left|\frac{s(b)-s_{n-1}(b)}{s_n(b)-s_{n-1}(b)}\right|$. *Then*

(*a*) *if* $l_s(a) < \infty$, $l_i(b) > 0$ *and* $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = 0$, *then* $\sum_{n=1}^{\infty} a_n$ *is fcst* $\sum_{n=1}^{\infty} b_n$

(*b*) *if* $s(a)-s_{n-1}(a) \neq 0$ *for all* $n \in \mathbb{N}$, $l_i(a) > 0$, $l_s(b) < \infty$ *and* $\sum_{n=1}^{\infty} a_n$ *is fcst* $\sum_{n=1}^{\infty} b_n$, *then* $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = 0$.

2 Main results

Lemma 4 *Let* $\sum_{n=1}^{\infty} b_n$ *be a convergent series such that* $s(b)-s_{n-1}(b) \neq 0$, $n \in \mathbb{N}$ *and let* $c \in \mathbb{R} \setminus \{0\}$. *The following are equivalent:*

(*a*) *there exist* $\sum_{n=1}^{\infty} a_n$ *fcst* $\sum_{n=1}^{\infty} b_n$ *such that* $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = c$,

(*b*) *there exists a convergent sequence of real numbers* $\{r_n\}_{n=1}^{\infty}$ *such that* $r_n \neq 0$, $n \in \mathbb{N}$, $\lim_{n\to\infty} r_n = r \neq 0$ *and* $\sum_{n=1}^{\infty} \frac{(s_n(b)-s_{n-1}(b))r_n}{s(b)-s_{n-1}(b)}$ *is a convergent series.*

Proof. (b) $\Rightarrow$ (a). Put $\varepsilon_n = \sum_{j=n}^{\infty} \frac{(B_j - B_{j+1})r_j c}{rB_j} + pB_n^3$ and $A_n = \varepsilon_n B_n$, $n \in \mathbb{N}$ where $B_n = s(b)-s_{n-1}(b)$, $n \in \mathbb{N}$. Let $p \in \mathbb{R} \setminus \{0\}$ be such that $A_n \neq A_{n+1}$ and $\varepsilon_n \neq 0$, $n \in \mathbb{N}$ (such p exists because the number of conditions for p is countable). Put $a_n = A_n - A_{n+1}$, $n \in \mathbb{N}$. It is clear that $\lim_{n\to\infty} A_n = 0$, $A_n \neq 0$, $n \in \mathbb{N}$, $\lim_{n\to\infty} \frac{s(a)-s_{n-1}(a)}{s(b)-s_{n-1}(b)} = \lim_{n\to\infty} \frac{A_n}{B_n} = 0$. Since

$$\frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = \frac{\varepsilon_n B_n - \varepsilon_{n+1}B_{n+1}}{B_n - B_{n+1}} = \varepsilon_{n+1} + \frac{B_n}{B_n - B_{n+1}}(\varepsilon_n - \varepsilon_{n+1}) = \tag{1}$$

$$= \left(\left(\frac{B_n - B_{n+1}}{rB_n}\right) r_n c + p\left(B_n^3 - B_{n+1}^3\right)\right) \frac{B_n}{B_n - B_{n+1}} + \varepsilon_{n+1},$$

we have that $\lim\limits_{n\to\infty}\frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)}=c$.

$(a)\Rightarrow(b)$. Put $\varepsilon_n=\frac{s(a)-s_{n-1}(a)}{s(b)-s_{n-1}(b)}$ and $B_n=s(b)-s_{n-1}(b)$, $n\in\mathbb{N}$. From (1) we have $\lim\limits_{n\to\infty}\frac{B_n}{B_n-B_{n+1}}(\varepsilon_n-\varepsilon_{n+1})=c\neq 0$. Hence there exists $n_0\in\mathbb{N}$ such that for every $n\geq n_0$, $\varepsilon_n-\varepsilon_{n+1}\neq 0$. Put $c_{n_0+n}=\frac{B_{n_0+n}}{B_{n_0+n}-B_{n_0+n+1}}(\varepsilon_{n_0+n}-\varepsilon_{n_0+n+1})$ $n\in\mathbb{N}$, then $c_{n_0+n}\neq 0$ and $\lim\limits_{n\to\infty}c_{n_0+n}=c$. From the previous equality for c_{n_0+n} we get $\varepsilon_{n_0+n+1}=\varepsilon_{n_0+1}-\sum\limits_{j=1}^{n}\frac{B_{n_0+j}-B_{n_0+j+1}}{B_{n_0+j}}c_{n_0+j}$. Since $\lim\limits_{n\to\infty}\varepsilon_n=0$ we have that $\sum\limits_{j=1}^{\infty}\frac{B_{n_0+j}-B_{n_0+j+1}}{B_{n_0+j}}c_{n_0+j}=\sum\limits_{j=1}^{\infty}\frac{b_{n_0+j}}{b_{n_0+j}+b_{n_0+j+1}+\dots}c_{n_0+j}$ is a convergent series. We define sequence $\{r_n\}_{n=1}^{\infty}$ as follows: $r_n=1$ if $n\leq n_0$ and $r_n=c_n$ if $n>n_0$, $n\in\mathbb{N}$. So $\sum\limits_{n=1}^{\infty}\left(\frac{s_n(b)-s_{n-1}(b)}{s(b)-s_{n-1}(b)}\right)r_n$ is a convergent series. $\square$

Lemma 5 *Let $\sum\limits_{n=1}^{\infty}b_n$ be a convergent series such that $s(b)-s_{n-1}(b)\neq 0$, $n\in\mathbb{N}$. The following are equivalent:*

(a) *there exists $\sum\limits_{n=1}^{\infty}a_n$ fcst $\sum\limits_{n=1}^{\infty}b_n$ such that* $\lim\limits_{n\to\infty}\dfrac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)}=+\infty$ $\left(\lim\limits_{n\to\infty}\dfrac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)}=-\infty\right)$,

(b) *there exists a sequence of real numbers $\{r_n\}_{n=1}^{\infty}$ such that $r_n\neq 0$, $n\in\mathbb{N}$,* $\lim\limits_{n\to\infty}r_n=+\infty$ $(\lim\limits_{n\to\infty}r_n=-\infty)$ *and* $\sum\limits_{n=1}^{\infty}\dfrac{(s_n(b)-s_{n-1}(b))r_n}{s(b)-s_{n-1}(b)}$ *is a convergent series.*

Proof. The proof of $(b)\Rightarrow(a)$ is similar to the proof of $(b)\Rightarrow(a)$ of Lemma 8. It is sufficient to put $\varepsilon_n=\sum\limits_{j=n}^{\infty}\frac{(B_j-B_{j+1})r_j}{B_j}+pB_n^3$, $n\in\mathbb{N}$. The proof of $(a)\Rightarrow(b)$ is similar to the proof of $(a)\Rightarrow(b)$ of Lemma 8. $\square$

Lemma 6 *Let $\sum\limits_{n=1}^{\infty}b_n$ be a convergent series such that $s(b)-s_{n-1}(b)\neq 0$, $n\in\mathbb{N}$. The following are equivalent:*

(a) *there exists $\sum\limits_{n=1}^{\infty}a_n$ fcst $\sum\limits_{n=1}^{\infty}b_n$ such that* $\lim\limits_{n\to\infty}\dfrac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)}$ *does not exist,*

(b) $\liminf\limits_{n\to\infty}\left|\dfrac{s_n(b)-s_{n-1}(b)}{s(b)-s_{n-1}(b)}\right|=0.$

Proof. $(a)\Rightarrow(b)$. Put $\gamma_n=\frac{B_n}{B_n-B_{n+1}}$, where $B_n=s(b)-s_{n-1}(b)$, $n\in\mathbb{N}$ and put $A_n=s(a)-s_{n-1}(a)$. Since $\lim\limits_{n\to\infty}\frac{s(a)-s_{n-1}(a)}{s(b)-s_{n-1}(b)}=\lim\limits_{n\to\infty}\frac{A_n}{B_n}=0$, $\frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)}=$

$\frac{A_n - A_{n+1}}{B_n - B_{n+1}} = \frac{A_{n+1}}{B_{n+1}} + \gamma_n\left(\frac{A_n}{B_n} - \frac{A_{n+1}}{B_{n+1}}\right)$ and $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)}$ does not exist, the sequence $\{\gamma_n\}_{n=1}^{\infty}$ is not bounded. Hence $\liminf_{n\to\infty}\left|\frac{1}{\gamma_n}\right| = \liminf_{n\to\infty}\left|\frac{s_n(b)-s_{n-1}(b)}{s(b)-s_{n-1}(b)}\right| = 0.$
$(b) \Rightarrow (a)$. Suppose that $\{\gamma_n\}_{n=1}^{\infty}$ is not bounded. One of the following cases holds:

I.) $\lim_{n\to\infty} \gamma_n = +\infty$,

II.) $\lim_{n\to\infty} \gamma_n = -\infty$,

III.) there are subsequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ of $\{\gamma_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \alpha_n = a \in \mathbb{R}$ and $\lim_{n\to\infty} \beta_n = +\infty$,

IV.) there are subsequences $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ of $\{\gamma_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} \alpha_n = a \in \mathbb{R}$ and $\lim_{n\to\infty} \beta_n = -\infty$.

I.) Let $\lim_{n\to\infty} \gamma_n = +\infty$. Suppose first that $\sum_{n=1}^{\infty} \frac{1}{\gamma_n} = +\infty$. We construct the sequence $\{r_n\}_{n=1}^{\infty}$, where $r_n \in \{-1, 1\}$, $n \in \mathbb{N}$ such that $\sum_{j=1}^{\infty} \frac{r_j}{\gamma_j}$ is a convergent series. By the induction we define an increasing sequence $\{n_m\}_{m=1}^{\infty}$, $n_m \in \mathbb{N}$. Choose any $s \in \mathbb{R}$, $s > \frac{1}{\gamma_1}$ and put $n_1 = \min\left\{n \in \mathbb{N}; \sum_{j=1}^{n} \frac{1}{\gamma_j} > s\right\}$, $n_2 = \min\left\{n > n_1; \sum_{j=1}^{n_1} \frac{1}{\gamma_j} - \sum_{j=n_1+1}^{n} \frac{1}{\gamma_j} < s\right\}$. Suppose that we have $n_1 < n_2 < ... < n_{m-1}$, $m > 2$. If $m = 2k+1$, $k \in \mathbb{N}$ we put $n_m = \min\{n > n_{2k}; \sum_{j=1}^{n_1} \frac{1}{\gamma_j} - \sum_{j=n_1+1}^{n_2} \frac{1}{\gamma_j} + ... - \sum_{j=n_{2k-1}+1}^{n_{2k}} \frac{1}{\gamma_j} + \sum_{j=n_{2k}+1}^{n} \frac{1}{\gamma_j} > s\}$, if $m = 2k+2$ we put
$n_m = \min\{n > n_{2k+1}; \sum_{j=1}^{n_1} \frac{1}{\gamma_j} - \sum_{j=n_1+1}^{n_2} \frac{1}{\gamma_j} + ... + \sum_{j=n_{2k}+1}^{n_{2k+1}} \frac{1}{\gamma_j} - \sum_{j=n_{2k+1}+1}^{n} \frac{1}{\gamma_j} < s\}$.
Define $\{r_n\}_{n=1}^{\infty}$ as follows: put $n_0 = 1$, then $r_n = (-1)^{m+1}$ if $n_{m-1} < n \le n_m$, $n \in \mathbb{N}$. Since $\sum_{n=1}^{\infty} \frac{1}{\gamma_n} = +\infty$ and $\lim_{n\to\infty} \frac{1}{\gamma_n} = 0$ we have that $\sum_{n=1}^{\infty} \frac{r_n}{\gamma_n}$ is a convergent series. Define $\varepsilon_n = \sum_{j=n}^{\infty} \frac{r_j}{\gamma_j} + pB_n^2$, $n \in \mathbb{N}$, where $p \in \mathbb{R}$, $p \neq 0$ such that $\varepsilon_n \neq 0$, $\varepsilon_n B_n \neq \varepsilon_{n+1} B_{n+1}$, $n \in \mathbb{N}$. (such p exists see proof of Lemma 8). Put $A_n = \varepsilon_n B_n$, $a_n = A_n - A_{n+1}$, $n \in \mathbb{N}$. Then $\lim_{n\to\infty} \frac{s(a)-s_{n-1}(a)}{s(b)-s_{n-1}(b)} = \lim_{n\to\infty} \frac{A_n}{B_n} = 0$ and since $\frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = \frac{A_n - A_{n+1}}{B_n - B_{n+1}} = \varepsilon_{n+1} + r_n + pB_n(B_n + B_{n+1})$, $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)}$ does not exists. If $\sum_{n=1}^{\infty} \frac{1}{\gamma_n}$ is a convergent series then $\sum_{n=1}^{\infty} \frac{(-1)^n}{\gamma_n}$ is again a convergent series because $\gamma_n > 0$ for $n \ge n_0$, $n_0 \in \mathbb{N}$. If we put $r_n = (-1)^n$, $n \in \mathbb{N}$, the proof is similar as in the case $\sum_{n=1}^{\infty} \frac{1}{\gamma_n} = +\infty$.
II.) The proof is similar as in I.)
III.) There exists a subsequence $\{\gamma_{n_k}\}_{k=1}^{\infty}$ of $\{\gamma_n\}_{n=1}^{\infty}$ such that $\lim_{k\to\infty} \gamma_{n_k} = +\infty$. Put $A = \{n_k; k \in \mathbb{N}\}$. We define $\{r_{n_k}\}_{k=1}^{\infty}$ by the behavior of $\sum_{k=1}^{\infty} \frac{1}{\gamma_{n_k}}$ as above.

Put

$$\varepsilon_{n+1} = \begin{cases} \varepsilon_n - \frac{r_n}{\gamma_n} - p(B_n^2 - B_{n+1}^2) & \text{for} \quad n > 1, n \in A \\ \varepsilon_n - p(B_n^2 - B_{n+1}^2) & \text{for } n > 1, n \notin A \end{cases}$$

where $\varepsilon_1 = \sum_{k=1}^{\infty} \frac{r_{n_k}}{\gamma_{n_k}} + pB_1^2$, p is determined as above and the proof is similar to the above part.
IV.) The proof is similar as in III.) □

It is easy to show that there exist convergent series $\sum_{n=1}^{\infty} b_n$ with $s(b) - s_{n-1}(b) \neq 0$, $n \in \mathbb{N}$ such that $\sum_{n=1}^{\infty} \frac{s_n(b)-s_{n-1}(b)}{s(b)-s_{n-1}(b)}$ is convergent series. There also exist convergent series $\sum_{n=1}^{\infty} b_n$, $s(b) - s_{n-1}(b) \neq 0$, $n \in \mathbb{N}$ and sequences $\{r_n\}_{n=1}^{\infty}$ such that $r_n \neq 0$, $n \in \mathbb{N}$, $\lim_{n\to\infty} r_n = +\infty$ ($\lim_{n\to\infty} r_n = -\infty$) and $\sum_{n=1}^{\infty} \frac{(s_n(b)-s_{n-1}(b))r_n}{s(b)-s_{n-1}(b)}$ is convergent series.

Proposition 7 *Let* $\sum_{n=1}^{\infty} a_n$ *be fcst* $\sum_{n=1}^{\infty} b_n$ *and let* $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = c \neq 0$. *Then* $\lim_{n\to\infty} \frac{s_n(b)-s_{n-1}(b)}{s(b)-s_{n-1}(b)} = 0$.

Proof. The proof follows from Lemma 8. □

Lemma 8 (*Lemma 2.2[4]*) *Let* $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ *be convergent real series with positive terms. Let* $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)}$ *exist. Then* $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = 0$ *if and only if* $\sum_{n=1}^{\infty} a_n$ *is fcst* $\sum_{n=1}^{\infty} b_n$.

Proposition 9 *Let* $\sum_{n=1}^{\infty} a_n$ *be fcst* $\sum_{n=1}^{\infty} b_n$ *and let* $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = c \neq 0$. *Then* $B_+ = \{n \in N; b_n > 0\}$, $B_- = \{n \in N; b_n < 0\}$, $A_+ = \{n \in N; a_n > 0\}$, $A_- = \{n \in N; a_n < 0\}$ *are infinite sets.*

Proof. By the way of contradiction we suppose that one of these sets is finite. Then there exists $n_0 \in \mathbb{N}$ such that $sign(a_n) = sign(a_m)$, and $sign(b_n) = sign(b_m)$ (where $sign(x) = 1$ if $x > 0$ and $sign(x) = -1$ if $x < 0$) for every $n, m > n_0$. From Lemma 2.2 [4] we get $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = 0$, a contradiction. □

Proposition 10 *Let* $\sum_{n=1}^{\infty} a_n$ *be fcst* $\sum_{n=1}^{\infty} b_n$ *and such that* $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)}$, $\lim_{n\to\infty} \frac{s_n(b)-s_{n-1}(b)}{s(b)-s_{n-1}(b)}$ *do not exist and* $\limsup_{n\to\infty} \left|\frac{s_n(a)-s_{n-1}(a)}{s(a)-s_{n-1}(a)}\right| < \infty$. *Then* $\liminf_{n\to\infty} \left|\frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)}\right| = 0$.

Proof. It follows from:
$\frac{s(a)-s_{n-1}(a)}{s(b)-s_{n-1}(b)} = \frac{s(a)-s_{n-1}(a)}{s_n(a)-s_{n-1}(a)} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} \frac{s_n(b)-s_{n-1}(b)}{s(b)-s_{n-1}(b)}$ and from Lemma 10. □

Proposition 11 *Let* $\sum_{n=1}^{\infty} b_n$ *be convergent series such that* $s(b) - s_{n-1}(b) \neq 0$, $n \in \mathbb{N}$ *and* $\liminf_{n\to\infty} \left|\frac{s_n(b)-s_{n-1}(b)}{s(b)-s_{n-1}(b)}\right| > 0$. *Let* $\sum_{n=1}^{\infty} a_n$ *be fcst* $\sum_{n=1}^{\infty} b_n$. *Then* $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = 0$.

Proof. The proof follows from Lemmas $9, 10, 11$. $\square$

Proposition 12 *Let* $\sum_{n=1}^{\infty} b_n$ *be a series such that* $b_n = q^n c_n$, $q \neq 0$, $|q| < 1$, $0 < k_1 < |c_n| < k_2$, $k_1, k_2 \in \mathbb{R}$, $s(b) - s_{n-1}(b) \neq 0$, $n \in \mathbb{N}$ *and let* $\sum_{n=1}^{\infty} a_n$ *be a fcst* $\sum_{n=1}^{\infty} b_n$. *Then* $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = 0$.

Proof. The inequality $|s(b)-s_{n-1}(b)| < k_2 \frac{|q|^n}{1-|q|}$ $n \in \mathbb{N}$ implies $\left|\frac{s_n(b)-s_{n-1}(b)}{s(b)-s_{n-1}(b)}\right| > \frac{k_1}{k_2}(1-|q|)$, $n \in \mathbb{N}$. So by Lemma 15 $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)} = 0$. $\square$

Lemma 13 *Let* $\sum_{n=1}^{\infty} b_n$ *be a convergent series such that* $s(b) - s_{n-1}(b) \neq 0$, $n \in \mathbb{N}$ *and* $\sum_{n=1}^{\infty} b_n = a$. *Let* $\lim_{n\to\infty} \left|\frac{s(b)-s_{n-1}(b)}{s_n(b)-s_{n-1}(b)}\right| = +\infty$. *Let* $\sum_{n=1}^{\infty} c_n$ *be a convergent series with a sum c. Let* $\limsup_{n\to\infty} \left|\frac{s_n(c)-s_{n-1}(c)}{s_n(b)-s_{n-1}(b)}\right| < \infty$. *Let* $\sum_{n=1}^{\infty} a_n$ *be a Kummer's transformation of* $\sum_{n=1}^{\infty} b_n$ *and* $\sum_{n=1}^{\infty} c_n$ *such that* $a_n \neq 0$ *for* $n \geq 2$ *and* $\limsup_{n\to\infty} \left|\frac{s(a)-s_{n-1}(a)}{s_n(a)-s_{n-1}(a)}\right| < \infty$. *Then* $\sum_{n=1}^{\infty} a_n = a$ *and* $\sum_{n=1}^{\infty} a_n$ *is a fcst* $\sum_{n=1}^{\infty} b_n$.

Proof. Put $A_n = s(a) - s_{n-1}(a)$, $B_n = s(b) - s_{n-1}(b)$, $n \in \mathbb{N}$. From $\frac{A_n}{B_n} = \frac{A_n - A_{n+1}}{B_n - B_{n+1}} \frac{\frac{A_n}{A_n - A_{n+1}}}{\frac{B_n}{B_n - B_{n+1}}}$ and from assumptions of the lemma it follows $\lim_{n\to\infty} \frac{A_n}{B_n} = 0$. $\square$

The above lemma is useful, for example in the case that the sum a is unknown and the sum c is known.

The next example shows that there exist Kummer's series $\sum_{n=1}^{\infty} a_n$ fcst $\sum_{n=1}^{\infty} b_n$ such that $\lim_{n\to\infty} \frac{s_n(a)-s_{n-1}(a)}{s_n(b)-s_{n-1}(b)}$ need not exist and the terms of both series need not be positive.

Example 14 *Let* $\sum_{n=1}^{\infty} b_n$ *be a convergent series such that* $b_1 > 0$, $b_{2n} = \frac{1}{4n^2+\sqrt{2n}}$, $b_{2n+1} = \frac{1}{4n^2-\sqrt{2n+1}}$, $n \in \mathbb{N}$. *Let* $\sum_{n=1}^{\infty} c_n$ *be a convergent series such that* $c_1 \in \mathbb{R} \setminus \{0\}$, $c_{2n} = \frac{-1}{2n^2(8n^3-1)}$, $c_{2n+1} = \frac{-1}{2n^2}$, $n \in \mathbb{N}$. *It is easy to see that* $\lim_{n\to\infty} \left|\frac{s(b)-s_{n-1}(b)}{s_n(b)-s_{n-1}(b)}\right| = +\infty$, $b_n \neq 0$, $s(b) - s_{n-1}(b) \neq 0$, $n \in \mathbb{N}$, $\limsup_{n\to\infty} \left|\frac{s_n(c)-s_{n-1}(c)}{s_n(b)-s_{n-1}(b)}\right| < +\infty$ *and*

$\lim_{n\to\infty} \frac{s_n(c)-s_{n-1}(c)}{s_n(b)-s_{n-1}(b)}$ *does not exists. Put* $c = \sum_{n=1}^{\infty} c_n$. *The series* $\sum_{n=1}^{\infty} a_n$ *such that* $a_n = b_n + c_n$, $n \geq 2$ *and* $a_1 = b_1 + c_1 - c$ *is a Kummer's series which fulfills* $\limsup_{n\to\infty} \left| \frac{s(a)-s_{n-1}(a)}{s_n(a)-s_{n-1}(a)} \right| < +\infty$, $s(a) - s_{2n-1}(a) > 0$ *and* $s(a) - s_{2n}(a) < 0$, *for* $n \geq 2$. *By the above lemma* $\sum_{n=1}^{\infty} a_n$ *is a fcst* $\sum_{n=1}^{\infty} b_n$ *and it has the same sum as* $\sum_{n=1}^{\infty} b_n$.

References

[1] Abramowitz M. and Stegun I.A.(eds.) (1972),Handbook of Mathematical Functions (National Bureau of Standards, Washington, D. C.).

[2] Bornemann F., Laurie D., Wagon S., and Waldvogel J. (2004), The SIAM 100-Digit Challenge: A Study in High-Accuracy Numerical Computing (Society of Industrial Applied Mathematics, Philadelphia).

[3] Brezinski C. (1977), Accélération de la Convergence en Analyse Numérique (Springer/Verlag, Berlin).

[4] Brezinski C. (1978), Algorithmes d'Accélération de la Convergence - Étude Numérique (Éditions Technip, Paris).

[5] Brezinski C. (1980), Padé-Type Approximation and General Orthogonal Polynomials (Birkhäuser, Basel).

[6] Brezinski C. (1991), History of Continued Fractions and Padé Approximants, (Springer/Verlag, Berlin).

[7] Brezinski C. (1991), A Bibliography on Continued Fractions, Padé Approximation. Sequence Transformation and Related Subjects (Prensas Universitarias de Zaragoza, Zaragoza) .

[8] Brezinski C. and Redivo Zaglia M. (1991), Extrapolation Methods (North-Holland, Amsterdam).

[9] Bromwich T.J.I. (1991), An Introduction to the Theory of Infinite Series (Chelsea, New York), 3rd edn. Originally published by Macmillan (London, 1908 and 1926).

[10] Caliceti E., Meyer-Hermann M., Ribeca P., Surzhykov A., and Jentschura U.D. (2007), From useful algorithms for slowly convergent series to physical predictions based on divergent perturbative expansions, Phys. Rep. 446, 1-96.

[11] Clark W.D., Gray H.L., and Adams J.E. (1969), A note on the T-transformation of Lubkin, J.Res.Natl.Bur.Stand. B73, 25-29.

[12] Cuyt A. and Wuytack L. (1987), Nonlinear Methods in Numerical Analysis (North-Holland, Amsterdam).

[13] D.F. Dawson, Matrix summability over certain classes of sequences ordered with respect to rate of convergence, Pacific Journal of Mathematics, vol. 24, no.1, 1968, pp. 51-56.

[14] Delahaye J.P. (1988), Sequence Transformations (Springer-Verlag, Berlin).
[15] Ferraro G. (2008), The Rise and Development of the Theory of Series up to the Early 1820s (Springer-Verlag, New York).
[16] Holý D., Matejíčka L., and Pinda Ľ.,(2008), On faster convergent infinite series, Int.J.Math.Math.Sci. 2008, 753632-1-753632-9.
[17] Homeier H.H.H. (2000), Scalar Levin-type sequence transformations, J.Comput.Appl.Math. 122,81-147. Reprinted as [17].
[18] Homeier H.H.H. (2000), Scalar Levin-type sequence transformations, in C. Brezinski(ed.), Numerical Analysis 2000, Vol. 2: Interpolation and Extrapolation, 81-147 (Elsevier, Amsterdam).
[19] Jentschura U.D., Mohr P.J., Soff G., and Weniger E.J. (1999), Convergence acceleration via combined nonlinear-condensation transformations, Comput. Phys. Commun. 116, 28-54.
[20] Keagy T.A., and Ford W.F. (1998), Acceleration by subsequence transformation, Pacific Journal of Mathematics, vol. 132, no.2, 1988, 357-362.
[21] Knopp K. (1964), Theorie und Anwendung der unendlichen Reihen (Springer-Verlag, Berlin).
[22] Liem C.B., Lü T., and Shih T.M. (1995), The Splitting Extrapolation Method (World Scientific, Singapore).
[23] Marchuk G.I., and Shaidurov V.V. (1983), Difference Methods and Their Extrapolations (Springer-Verlag, New York).
[24] Press W.H., Teukolsky S.A., Vetterling W.T., and Flannery B.P. (2007), Numerical Recipes: The Art of Scientific Computing (Cambridge U. P., Cambridge).
[25] Salzer H.E. (1955), A simple method for summing certain slowly convergent series, Journal of Mathematics and Physics, vol. 33, 1955, 356-359.
[26] Sidi A. (2003), Practical Extrapolation Methods (Cambridge U.P., Cambridge).
[27] Smith D.A., and Ford W.F. (1979), Acceleration of linear and logarithmic convergence, Siam Journal of Numerical Analysis, vol. 16, no.2, 1979, 223-240.
[28] Srivastava H.M. and Choi J. (2001), Series Associated with the Zeta and Related Functions (Kluwer, Dordrecht).
[29] Šalát T. (1974), Nekonečné rady, ACADEMIA nakladatelství Československé akademie věd Praha, (1974).
[30] Tripathy B.C., and Sen M. (2005), A note on rate of convergence of sequences and density of subsets of natural numbers, Italian Journal of Pure and Applied Mathematics, vol. 17, 2005, 151-158.
[31] Walz G. (1996), Asymptotics and Extrapolation (Akademie Verlag, Berlin).
[32] Weniger E.J. (1989), Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series, Comput. Phys. Rep. 10, 189-371, Los Alamos Preprint math-ph/0306302(http://arXiv.org).

[33] Wimp J. (1981), Sequence Transformations and Their Applications (Academic Press, New York).
[34] Wimp J. (1972), Some transformations of monotone sequences, Mathematics of Computation, vol. 26, no.117, 1972, 251-254.